# CHAPTER

16

PROBABILITY AND THE BINOMIAL THEOREM

Medical facilities and pharmaceutical companies engage in research to develop new ways to cure or prevent disease and to ease pain. Often after years of experimentation a new drug or therapeutic technique will be developed. But before this new drug or technique can be accepted for general use, it is necessary to determine its effectiveness and the possible side effects by studying data from an experimental group of people who use the new discovery. The principles of statistics provide the guidelines for choosing such a group, for conducting the necessary study, and for evaluating the results.

This chapter will review some of the basic principles of probability and extend these principles to questions such as those above.

16-1 THE COUNTING PRINCIPLE

Tossing a coin is familiar to everyone. We say that when a fair coin is tossed, either side, heads or tails, is equally likely to appear face up. If a penny and a nickel are tossed, each coin can show heads, H, or tails, T. Each result is called an **outcome**. If we designate the outcomes on the penny as H and T and the outcomes on the nickel as H and T, then the set of possible outcomes for tossing the two coins contains four pairs:

$$\{(H, H), (H, T), (T, H), (T, T)\}$$
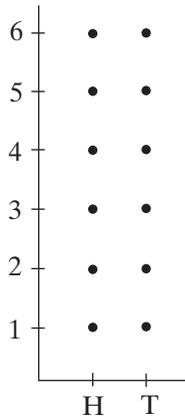
This set of all possible outcomes is the **sample space** for the activity.

If a nickel and a die are tossed, there are 2 ways in which the nickel can land and 6 ways in which the die can land. There are 2×6 or 12 possible outcomes. The sample space is shown as a graph of ordered pairs to the left and as a set of ordered pairs below:

$$\{(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6), (H, 1), (H, 2),$$
$$(H, 3), (H, 4), (H, 5), (H, 6)\}$$

These examples illustrate the **Counting Principle**.

► *The Counting Principle***: If one activity can occur in any of *m* ways and a second activity can occur in any of *n* ways, then both activities can occur in the order given in *m · n* ways.**

The counting principle can be applied to any number of activities. For example, consider a set of three cards lying facedown numbered 4, 6, and 8. Cards are drawn, one at a time, and not replaced. The numbers are used in the order in which they are drawn to form three-digit numbers. Note that since each draw comes from the same set of cards and the card or cards previously drawn are not replaced, the number of cards available decreases by 1 after each draw.

The table below represents the number of ways the cards may be drawn. However, it does *not* show us what those cards may be. We can draw a **tree diagram** to represent the outcomes.

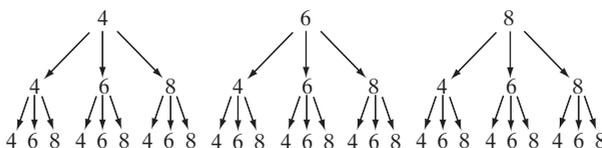| Draw	Number of Cards	Number of Outcomes
1st	3	3
2nd	2	$3 \times 2 = 6$
3rd	1	$3 \times 2 \times 1 = 6$
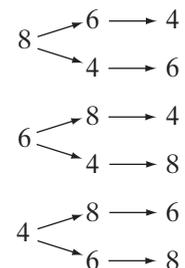
- There are 6 outcomes in the sample space: {468, 486, 684, 648, 846, 864}.
- The outcomes greater than 800 are the elements of the subset {846, 864}.
- The outcomes less than 650 are the elements of the subset {468, 486, 648}.
- The outcomes that are odd numbers are the elements of the subset { } or ∅, the empty set.
- The outcomes that are even numbers are the elements of the subset {468, 486, 684, 648, 846, 864}, that is, the sample space.

A subset of the sample space is an **event**. In the example above, cards were drawn without replacement. The outcome of each draw depended on the previous, so the draws are **dependent events**.

Consider a similar example. There are three cards lying facedown numbered 4, 6, and 8. Three cards are drawn, one at a time, and replaced after each draw. The numbers are used in the order in which they are drawn to form three-digit numbers. Note that since each card previously drawn is replaced, the number of cards available is the same for each draw.

Draw	Number of Cards	Number of Outcomes
1st	3	3
2nd	3	$3 \times 3 = 9$
3rd	3	$3 \times 3 \times 3 = 27$

The number of possible outcomes when the drawn card is replaced and can be drawn again is significantly greater than the number of outcomes in the similar example without replacement. Since the outcome of each draw does *not* depend on the previous, we call the draws **independent events**.

EXAMPLE I

In a debate club, there are 12 boys and 15 girls.

a. In how many ways can a boy and a girl be selected to represent the club at a meeting?

b. Are the selection of a boy and the selection of a girl dependent or independent events?

Solution **a.** Use the counting principle. There are 12 possible boys and 15 possible girls who can be selected. There are 12×15 or 180 possible pairs of a boy and a girl that can be selected.

b. The selection of a boy does not affect the selection of a girl, and vice versa. Therefore, these are independent events. ☐

EXAMPLE 2

The skating club is holding its annual competition at which a gold, a silver, and a bronze medal are awarded.

a. In how many different ways can the medals be awarded if 12 skaters are participating in the competition?

b. Are the awarding of medals dependent or independent events?

Solution **a.** • The gold medal can be awarded to one of the 12 skaters.

• After the gold medal is awarded, the silver medal can be awarded to one of 11 skaters.

• After the silver medal is awarded, the bronze medal can be awarded to one of 10 skaters.

There are $12 \times 11 \times 10$ or 1,320 possible ways the medals can be awarded.

b. Since the silver medalist cannot have won the gold and the bronze medalist cannot have won either the gold or silver, the awarding of medals are dependent events. ☐

Exercises

Writing About Mathematics

1. How is choosing a boy and a girl from 12 boys and 12 girls to represent a club different from choosing two girls from 12 girls to be president and treasurer of the club?

2. Compare the number of ordered pairs of cards that can be drawn from a deck of 52 cards without replacement to the number of ordered pairs of cards that can be drawn from a deck of 52 cards with replacement.

Developing Skills

In 3–6, for the given values of r and n, find the number of ordered selections of r objects from a collection of n objects without replacement.

3. $r = 4, n = 4$ **4.** $r = 4, n = 5$ **5.** $r = 3, n = 8$ **6.** $r = 2, n = 12$

In 7–10, for the given values of r and n, find the number of ordered selections of r objects from a collection of n objects with replacement.

7. $r = 4, n = 4$ **8.** $r = 4, n = 5$ **9.** $r = 2, n = 8$ **10.** $r = 3, n = 5$

11. For the sample space {A, B, C, D}, determine how many events are possible.

In 12–17, state whether the events are independent or dependent.

12. Tossing a coin and rolling a die

13. Selecting a winner and a runner-up from 7 contestants

14. Picking 2 cards from a standard deck without replacing the first card

15. Buying a magazine and a snack for a train trip

16. Choosing a size and a color for a sweater

17. Writing a 5-letter word using the alphabet if letters cannot be repeated

In 18–27, determine the number of possible outcomes.

18. Tossing a die 3 times   **19.** Tossing a coin 5 times

20. Choosing breakfast of juice, cereal, and fruit from 3 juices, 5 cereals, and 4 fruits

21. Assembling an outfit from 6 shirts, 4 pairs of pants, and 2 pairs of shoes

22. Choosing a 4-digit entry code using 0–9 if digits cannot be repeated

23. Choosing from 5 flavors of iced tea in 3 different sizes with or without sugar

24. Ordering an ice cream cone from a choice of 31 flavors, 3 types of cone, with or without one of 4 toppings

25. Making a 7-character license plate using the letters of the alphabet and the digits 1–4 if the first three characters must be non-repeating letters and the remaining four are digits that may repeat

26. Seating Andy, Brenda, Carlos, Dabeed, and Eileen in a row of 5 seats

27. Seating Andy, Brenda, Carlos, Dabeed, and Eileen in a row of 5 seats with Andy and Eileen occupying end seats

Applying Skills

28. In how many ways can 5 new bus passengers choose seats if there are 8 empty seats?

29. In how many ways can 3 different job openings be filled if there are 8 applicants?

30. In how many ways can 5 roles in the school play be filled if there are 9 possible people trying out? (Each role is to be played by a different person.)

31. From the set {2, 3, 4, 5, 6}, a number is drawn and replaced. Then a second number is drawn. How many two-digit numbers can be formed?

32. In a class of 22 students, the teacher calls on a student to give the answer to the first homework problem and then calls on a student to give the answer to the second homework problem.

    a. How many possible choices could the teacher have made if the same student was not called on twice?

    b. How many possible choices could the teacher have made if the same student may have been called on twice?

33. A manufacturer produces jeans in 9 sizes, 7 different shades of blue, and 6 different leg widths. If a branch store manager orders two pairs of each possible type, how many pairs of jeans will be in stock?

34. A restaurant offers a special in which a diner can have a choice of appetizer, entree, and dessert for $11.99.

Appetizer	Entree	Dessert
Mozzarella sticks	Roast chicken	Chocolate pie
Garden salad	Steamed fish	Apple pie
Tomato soup	Roast beef	Cherry pie
	Vegetable curry	Blueberry pie
		Strawberry-rhubarb pie
		Lemon meringue pie

   a. How many different meals can a diner order?

   b. How many of these choices contain vegetable curry?

   c. If a diner does not want to eat the soup or the fish, how many different meals can she order?

35. On a certain ski run, there are 8 places where a skier can choose to go to the left or to the right. In how many different ways can the skier cover the run?

36. Stan has letter tiles $\boxed{A}\ \boxed{M}\ \boxed{T}\ \boxed{E}$.

   a. How many different ways can Stan arrange all four tiles?

   b. How many different arrangements begin with a vowel?

37. Six students will be seated in a row in the classroom.

   a. How many different ways can they be seated?

   b. If one student forgot his eyeglasses and must occupy the front seat, how many different seatings are possible?

38. To duplicate a key, a locksmith begins with a dummy key that has several sections. The locksmith grinds a specific pattern into each section.

   a. A particular brand of house key includes 6 sections, and there are 4 possible patterns for each section. How many different house keys are possible?

   b. A desk key has 3 sections, and 64 different keys are possible. How many patterns are available for each section if each section has the same number of possible patterns?

39. A physicist, a chemist, a biologist, an astronomer, a geologist, and a mathematician are guest speakers at a government-sponsored forum on scientific research in the twenty-first century. The speakers will be seated in a row on a raised platform at the front of the meeting room.

   a. How many different ways can the speakers be seated?

   b. The astronomer and the mathematician have co-written a paper that they will present. How many ways can the speakers be seated if the astronomer and the mathematician wish to sit side-by-side?

40. A given region has the telephone prefix 472.

   a. How many 7-digit telephone numbers are possible with this prefix?

   b. How many of these possibilities end with 4 unique digits?

   c. How many of these possibilities form an even number?

41. How many 2-letter patterns can be formed from the alphabet if the first letter is a vowel (A, E, I, O, U) and the second letter is a consonant that occurs later in the alphabet than the vowel in the first position?

42. How many 2-letter patterns may be formed from the alphabet if the first letter is a consonant and the second letter is a vowel that occurs earlier than the first letter?

16-2 PERMUTATIONS AND COMBINATIONS

Permutations

A **permutation** is an arrangement of objects in a specific order. For example, if four students are scheduled to give a report in class, then each possible order in which the students give their reports is a permutation. The number of possible permutations is shown in the following table.

	Number of Students	Number of Outcomes
1st	4	4
2nd	3	$4 \cdot 3 = 12$
3rd	2	$4 \cdot 3 \cdot 2 = 24$
4th	1	$4 \cdot 3 \cdot 2 \cdot 1 = 24$

We say that this is *the number of permutations of 4 things taken 4 at a time*, which can be symbolized as $_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. There are 24 permutations; that is, there are 24 different orders in which the 4 students can give their reports.

We know that for any natural number n, n factorial is the product of the natural number from n to 1. Recall that the symbol for n factorial is $n!$:

$$n! = n(n - 1)(n - 2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

Therefore, the number of ways 4 students can give their reports is:

$$_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

▶ **For any positive integer n, the number of arrangements of n objects taken n at a time can be represented as:**

$$_nP_n = n!$$

If there are 20 students in the class and any 4 of them may be asked to give a report today, then this permutation includes both a selection of 4 of the 20 students and the arrangement of those 4 in some order. In this case, there are 20 possible choices for the first report, 19 for the second, 18 for the third and 17 for the fourth.

	Number of Students	Number of Outcomes
1st	20	20
2nd	19	20×19
3rd	18	$20 \times 19 \times 18$
4th	17	$20 \times 19 \times 18 \times 17$

The number of orders in which 4 of the 20 students can give a report is the number of permutations of 20 things taken 4 at a time. This permutation is symbolized as:

$$_{20}P_4 = 20 \times 19 \times 18 \times 17$$

Note that the last factor is $(20 - 4 + 1)$.

We can think of $20 \cdot 19 \cdot 18 \cdot 17$ as $20!$ with the last 16 factors canceled.

$$\frac{20!}{16!} = \frac{20 \times 19 \times 18 \times 17 \times \cancel{16 \times 15 \times \cdots \times 3 \times 2 \times 1}^{1}}{\cancel{16 \times 15 \times \cdots \times 3 \times 2 \times 1}_{1}}$$

$$= 20 \times 19 \times 18 \times 17 = {_{20}P_4}$$

▶ **In general, for $r \leq n$, the number of permutations of n things taken r at a time is:**

$$_nP_r = n(n - 1)(n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$$

EXAMPLE I

In how many different ways can the letters of the word PENCIL be arranged?

Solution   This is the number of permutations of 6 things taken 6 at a time.

$$_6P_6 = 6!$$

Use a calculator to evaluate 6!. Factorial is entry 4 of the PRB menu located under MATH .

ENTER: 6 | MATH | ◀ | 4 | | ENTER |

DISPLAY:

```
6!
            720
```

Alternative   The value of $_6P_6$ can also be found directly by using the *nPr* function of the
Solution   calculator. The *nPr* function is entry 2 of the PRB menu located under MATH .
The value of *n* is entered before accessing this symbol.

ENTER: 6 | MATH | ◀ | 2 | 6 | ENTER |

DISPLAY:

```
6 nPr 6
            720
```

Answer   6! = 720 ways

EXAMPLE 2

In how many different ways can the letters of the word PENCIL be arranged
if the first letter must be a consonant?

Solution   Since the first letter must be a consonant and there are 4 consonants, there are
4 possible choices for the first letter. Then, after the first letter has been cho-
sen, there are 5 letters to be arranged.

$$4 \times {_5P_5} = 4 \times 5! = 4 \times 5 \times 4 \times 3 \times 2 \times 1$$

Use the calculator to evaluate $4 \times 5!$.	Or use the calculator to evaluate $4 \times {}_5P_5$.
ENTER: 4 [×] 5 [MATH] [◄]	ENTER: 4 [×] 5 [MATH] [◄] [2]
[4] [ENTER]	5 [ENTER]

DISPLAY:	DISPLAY:
4*5! 480	4*5 nPr 5 480

Answer 480 ways

Permutations with Repetition

In the two examples given above, there are 6 different letters to be arranged. Would the problem have been different if the word had been ELEVEN? This is again a 6-letter word, but 3 of the letters are the same. Begin by thinking of the E's as different letters: E_1, E_2, and E_3. There are 480 arrangements of the letters. Consider all the arrangements in which the consonants are in given places. For example:

$$E_1LE_2VE_3N \qquad E_2LE_1VE_3N \qquad E_3LE_2VE_1N$$
$$E_1LE_3VE_2N \qquad E_2LE_3VE_1N \qquad E_3LE_1VE_2N$$

For every arrangement in which L, V, and N are fixed, there are 3! or 6 different arrangements of the E's. Divide the number of arrangements of all of the letters by the number of arrangements of the E's.

The number of arrangements of the 6 letters in which 3 of them are alike is:

$$\frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{720}{6} = 120$$

We can think of this as placing just the 3 consonants in the 6 available positions. There are 6 possible positions for the first consonant, 5 possible positions for the second consonant, and 4 possible positions for the third consonant.

The number of arrangements of the 6 letters in which 3 of them are alike is:

$$6 \times 5 \times 4 = 120$$

► **In general, the number of permutations of *n* things taken *n* at a time when *a* are identical is:**

$$\frac{n!}{a!}$$

EXAMPLE 3

Helene is lining up beads to plan a necklace. She has a total of 36 beads and 32 of them are identical. How many different arrangements of the 36 beads can she make?

Solution   The number of arrangements of 36 beads with 32 identical beads is $\frac{36!}{32!}$.

Use the calculator to evaluate $\frac{36!}{32!}$.

ENTER: 36 MATH   ◀   4   ÷   32 MATH   ◀   4   ENTER

DISPLAY:
```
36!/32!
              1413720
```

Answer   1,413,720 arrangements

For some permutations of n things taken n at a time, more than one thing repeats. Let the number of times each of r things repeats be $a_1, a_2, \ldots, a_r$. Then we can say that the number of permutations of n things taken n at a time with multiple repetitions is:

$$\frac{n!}{a_1! \cdot a_2! \cdot \cdots \cdot a_r!}$$

EXAMPLE 4

A music teacher is arranging a recital for her students. There are 7 students, each with his or her own instrument: 3 play the piano, 2 play the violin, 1 plays the flute, and 1 plays the cello. In how many ways can the order of the instruments be arranged?

Solution   This is an arrangement of 7 instruments with 3 alike and 2 alike. Divide the number of arrangements of 7 instruments by the number arrangements of the 3 pianos and by the number arrangements of the 2 violins.

$$\frac{7!}{3! \times 2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (2 \times 1)} = 7 \times 6 \times 5 \times 2 = 420 \ \textit{Answer}$$

EXAMPLE 5

The number of arrangements of 5 letters from a given set of letters is 12 times the number of arrangements of 3 letters from the set. How many letters are in the set?

Solution Let x be the number of letters in the set.

$$_xP_5 = 12(_xP_3)$$

$$\frac{x!}{(x-5)!} = 12\left(\frac{x!}{(x-3!)}\right)$$

$$\frac{x!}{(x-5)!}\left[\frac{(x-3)!}{x!}\right] = 12$$

$$\frac{x!}{x!}\left[\frac{(x-3)!}{(x-5)!}\right] = 12$$

$$1\left[\frac{(x-3)(x-4)\cancel{(x-5)}\cancel{(x-6)}\cdots\cancel{(3)(2)(1)}}{\cancel{(x-5)}\cancel{(x-6)}\cdots\cancel{(3)(2)(1)}}\right] = 12$$

$$(x-3)(x-4) = 12$$

$$x^2 - 7x + 12 = 12$$

$$x^2 - 7x = 0$$

$$x(x-7) = 0$$

$$x = 0 \quad \text{or} \quad x = 7$$

There cannot be 0 letters. Therefore, there are 7 letters in the set. *Answer*

Combinations

Five members of a club, Adam, Brian, Celia, Donna, and Elaine, have volunteered to represent the club at a competition, but the club can send only two members. The names of the volunteers are put in a box and two are drawn at random.

The first column of the table to the right shows all possible orders in which two names can be drawn. Each pair of names occurs in two different orders. The second column lists all possible pairs of names when order is not important.

Selection	Teams
(A, B), (B, A)	{A, B}
(A, C), (C, A)	{A, C}
(A, D), (D, A)	{A, D}
(A, E), (E, A)	{A, E}
(B, C), (C, B)	{B, C}
(B, D), (D, B)	{B, D}
(B, E), (E, B)	{B, E}
(C, D), (D, C)	{C, D}
(C, E), (E, C)	{C, E}
(D, E), (E, D)	{D, E}

Each ordered pair is a permutation. Each element of a set in which order is *not* important is a **combination**. The number of combinations of n things taken r at a time is symbolized $_nC_r$ or as $\binom{n}{r}$.

In order to find the number of combinations, we must first find the number of permutations and then divide that number by the number of rearrangements of the selected objects among themselves.

▶ **In general, for $r \leq n$, the number of combinations of n things taken r at a time is:**

$$_nC_r = \frac{_nP_r}{_rP_r} \quad \text{or} \quad \binom{n}{r} = \frac{_nP_r}{r!}$$

In the example above, we can say that the number of teams of two club members chosen from a group of five volunteers is:

$$_5C_2 = \frac{_5P_2}{_2P_2} = \frac{5 \times 4}{2 \times 1} = 10$$

On a calculator, we can find the number of combinations by using keys similar to those we used for permutations. The symbol nCr is entry 3 of the PRB menu located under MATH . To evaluate $_5C_2$:

ENTER: 5 MATH   ◄   3   2 ENTER

DISPLAY:
```
5 nCR 2
                10
```

The calculator displays the answer, 10.

EXAMPLE 6

How many different combinations of five letters can be selected from the alphabet?

Solution   This is a combination of 26 things taken 5 at a time.

$$_{26}C_5 = \frac{26 \times 25 \times 24 \times 23 \times 22}{5 \times 4 \times 3 \times 2 \times 1}$$

Use a calculator to evaluate.

ENTER: 26 MATH   ◄   3   5 ENTER

DISPLAY:
```
26 nCR 5
              65780
```

Answer   65,780

EXAMPLE 7

How many different combinations of 5 letters can be drawn from the alphabet if 3 are consonants and 2 are vowels?

Solution   There are 21 consonants and 5 vowels in the alphabet.

First find the number of combinations of 3 out of the 21 consonants.

$$_{21}C_3 = \frac{21 \times 20 \times 19}{3 \times 2 \times 1} = 1,330$$

Then find the number of combinations of 2 out of 5 vowels.

$$_5C_2 = \tfrac{5 \times 4}{2 \times 1} = 10$$

Now use the counting principle: $1{,}330 \times 10 = 13{,}300$ *Answer*

Exercises

Writing About Mathematics

1. Show that $_nC_r = \dfrac{n!}{(n - r)! \times r!}$.

2. Show that $n! = n(n - 1)!$.

Developing Skills

In 3–22, evaluate each expression.

3. $5!$ **4.** $12!$ **5.** $8! \div 3!$ **6.** $9! \div 8!$

7. $\dfrac{10!}{3!}$ **8.** $_6P_6$ **9.** $_8P_4$ **10.** $_6P_5$

11. $_5P_2$ **12.** $_4C_3$ **13.** $_{12}C_7$ **14.** $_{12}C_5$

15. $_{10}P_4 \div 4!$ **16.** $\binom{15}{5}$ **17.** $\binom{15}{10}$ **18.** $\binom{4}{4}$

19. $\binom{12}{0}$ **20.** $_5P_2 \times _3P_2$ **21.** $_8C_3 \div _8C_5$ **22.** $_{20}C_5 \div _{20}C_{15}$

In 23–30, find the number of different arrangements that are possible for the letters of each of the following words.

23. FACTOR   **24.** DIVIDE   **25.** EXCEED   **26.** ABSCISSA

27. RANDOMLY   **28.** REMEMBERED   **29.** STATISTICS   **30.** MATHEMATICS

31. A box contains 9 red, 4 blue, and 6 yellow chips. In how many ways can 6 chips be chosen if:

    a. all 6 chips are red?   **b.** all 6 chips are yellow?

    c. 2 chips are blue?   **d.** 3 chips are red?

    e. 4 chips are yellow?   **f.** there are 2 chips of each color?

    g. 3 chips are red and 3 chips are blue?   **h.** 5 chips are red and 1 chip is yellow?

In 32–37, determine the number of different arrangements.

32. The finishing order of 7 runners in a race

33. Seating of 5 students in a row of 9 chairs

34. Stacking 6 red, 4 yellow, and 2 blue t-shirts

35. The order in which a student answers 8 out of 10 test questions

36. Making a row of coins using 4 pennies, 3 nickels, and 3 dimes

37. Rating 6 employees in order of their friendliness

In 38–43, solve for x.

38. $_6P_2 = x(_5P_3)$

39. $_xP_6 = 30(_xP_4)$

40. $_{13}P_5 = 1{,}287(_xP_x)$

41. $_xC_6 = _xC_4$

42. $\binom{x}{8} = \binom{x}{7}$

43. $_{12}C_4 = _xC_8$

Applying Skills

44. Show that $_nC_r = _nC_{n-r}$.

45. At the library, Jordan selects 8 books that he would like to read but decides to check out just 5 of them. How many different selections can he make?

46. Eli has homework assignments for 5 subjects but decides to complete 4 of them today and complete the fifth before class tomorrow. In how many different orders can he choose 4 of the 5 assignments to complete today?

47. There are 12 boys and 13 girls assigned to a class. In how many ways can 2 boys and 3 girls be selected to transfer to a different class?

48. In how many ways can the letters of CIRCLE be arranged if the first and last must be consonants?

49. In how many ways can 6 transfer students be assigned to seats in a classroom that has 10 empty seats?

50. From a standard deck of 52 cards, how many hands of 5 cards can be dealt?

51. A local convenience store hires three students to work after school. Next month, there are 20 days on which they will work. Alex will work 8 days, Rosa will work 6 days, and Carla will work 6 days. In how many ways can their schedule for the month be arranged?

52. Marie has an English assignment and a history assignment that require research in the library. She makes a schedule for the 5 days of this week to work on the English assignment for 2 days and the history assignment for 3 days. How many different schedules are possible?

53. Roy and Valerie are playing a game of tic-tac-toe. In how many different ways can the first 4 moves in the game be made?

54. Rafael is running for mayor of his town. He sent out a survey asking his constituents to rank the following issues in order of importance to them: crime, unemployment, air quality, schools, public transportation, parking, and taxes.

a. How many different rankings are possible?

b. What fraction of the possible rankings put crime first?

c. How many of the possible rankings are in alphabetical order?

55. The Electronic Depot received 108 mp3 players, 12 of which are defective. In how many different ways can 6 players be selected for display such that:

a. none are defective?

b. half are defective?

c. all are defective?

56. The program director at the local sports center wants to schedule the following classes on Wednesdays: aerobics, tai chi, yoga, modern dance, fitness, and weight training. Starting at 9:00 A.M., each class begins on the hour and ends at 10 minutes before the hour.

a. How many different ways can the classes be scheduled?

b. If the yoga instructor is available only for a 9:00 A.M. class, how many different ways can the classes be scheduled?

57. A medical researcher has received approval to test a new combination drug therapy. A total of 6 doses of drug X, 3 doses of drug Y, and 4 doses of drug Z are to be given on successive days. The researcher wants to determine if some orders are more effective than others. How many different orders are possible?

16-3 PROBABILITY

Probability is an estimate of the frequency of an outcome in repeated trials. **Theoretical probability** is the ratio of the number of elements in an event to the number of equally likely elements in the sample space. For example, the sample space for tossing two coins, $\{(H, H), (H, T), (T, H), (T, T)\}$, has four equally likely elements and the event of landing two heads, $\{(H, H)\}$, has one element. The theoretical probability that two heads will show when two coins are tossed is $\frac{1}{4}$.

Let $P(E)$ = the probability of an event E

$n(E)$ = the number of element in event E

$n(S)$ = the number of elements in the sample space

$$P(E) = \frac{n(E)}{n(S)}$$

An event can be any subset of the sample space, including the empty set and the sample space itself: $0 \leq n(E) \leq n(S)$. Therefore, since $n(S)$ is a positive number,

$$\frac{0}{n(S)} \leq \frac{n(E)}{n(S)} \leq \frac{n(S)}{n(S)} \quad \text{or} \quad 0 \leq P(E) \leq 1$$

• $P(E) = 0$ when the event is the empty set, that is, there are no outcomes.

• $P(E) = 1$ when the event is the sample space, that is, certainty.

• $P(E) + P(\text{not } E) = 1$

For example, when a die is tossed, the probability of the die showing a number greater than 7 is 0 and the probability of the die showing a number less than 7 is 1. The probability of the die showing 2 is $\frac{1}{6}$ and the probability of a die showing a number that is not 2 is $\frac{5}{6}$.

$$P(2) + P(\text{not } 2) = \tfrac{1}{6} + \tfrac{5}{6} = \tfrac{6}{6} = 1$$

Some probabilities are arrived at by considering observed data or the results of an experiment or simulation. **Empirical probability** or **experimental probability** is the best estimate, based on a large number of trials, of the ratio of the number of successful results to the number trials. If we toss two coins four times, we cannot expect that two heads will occur exactly once. However, if this experiment, tossing two coins, is repeated thousands of times, the empirical probability, that is the ratio of the number of times two heads appear to the number of times the coins are tossed, will be approximately $\frac{1}{4}$.

Permutations, Combinations, and Probability

There are 8 boys and 12 girls in a chess club. Two boys and two girls are to be selected at random to represent the club at a tournament. What is the probability that Jacob and Emily will both be chosen? We can answer this question by using combinations.

There are $\frac{8 \times 7}{2 \times 1}$ or 28 possible combinations of 2 boys and $\frac{12 \times 11}{2 \times 1}$ or 66 possible combinations of 2 girls. Therefore there are 28×66 or 1,848 possible choices: $n(S) = 1,848$.

If Jacob is chosen, the pair of boys can be Jacob and one of the 7 other boys. Therefore, there are 7 pairs that include Jacob.

If Emily is chosen, the pair of girls can be Emily and one of the 11 other girls. Therefore, there are 11 pairs that include Emily.

- There are 7×11 or 77 choices that include Jacob and Emily: $n(E) = 77$.
- The probability that Jacob and Emily are chosen is $\frac{77}{1,848} = \frac{1}{24}$.

  We can also determine this probability in another way.

- The probability the Jacob is chosen is $\frac{7}{28} = \frac{1}{4}$.
- The probability that Emily is chosen is $\frac{11}{66} = \frac{1}{6}$.
- The probability that both Jacob and Emily are chosen is $\frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$.

There are 4 possible results of this choice: both Jacob and Emily are chosen, Jacob is chosen but not Emily, Emily is chosen but not Jacob, and neither Jacob nor Emily is chosen.

$$P(\text{Jacob is chosen}) = \tfrac{1}{4} \qquad P(\text{Jacob is not chosen}) = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

$$P(\text{Emily is chosen}) = \tfrac{1}{6} \qquad P(\text{Emily is not chosen}) = 1 - \tfrac{1}{6} = \tfrac{5}{6}$$

- $P(\text{Jacob and Emily are chosen}) = \tfrac{1}{4} \times \tfrac{1}{6} = \tfrac{1}{24}$

- $P(\text{Jacob is chosen but not Emily}) = \tfrac{1}{4} \times \tfrac{5}{6} = \tfrac{5}{24}$

- $P(\text{Emily is chosen but not Jacob}) = \tfrac{1}{6} \times \tfrac{3}{4} = \tfrac{3}{24}$

- $P(\text{neither Jacob nor Emily is chosen}) = \tfrac{3}{4} \times \tfrac{5}{6} = \tfrac{15}{24}$

The sum of all possible probabilities must be 1. To check:

$$\tfrac{1}{24} + \tfrac{5}{24} + \tfrac{3}{24} + \tfrac{15}{24} = \tfrac{24}{24} = 1 ✔$$

In the solution of this problem we have used the following relationship:

▶ **If $P(E) = p$, then, $P(\text{not } E) = 1 - p$.**

We have also used the probability form of the Counting Principle:

▶ **If E_1 and E_2 are independent events, and if $P(E_1) = p$ and $P(E_2) = q$, then the probability of both E_1 and E_2 occurring is $p \times q$.**

EXAMPLE 1

The letters of the word CABIN are rearranged at random. What is the theoretical probability that one arrangement chosen at random will begin and end with a vowel?

Solution The sample space has 5! elements: $n(S) = 120$.

Since the first and last letters must be vowels, there are 2 possible letters for the first place and 1 possible letter for the last. There are 3! permutations of the remaining three letters: $n(E) = 2 \times 3 \times 2 \times 1 \times 1 = 12$.

$$P(E) = \frac{n(E)}{n(S)} = \frac{12}{120} = \frac{1}{10} \ \textit{Answer}$$

EXAMPLE 2

The student attendance record for one semester is shown on page 690. If two students are chosen at random, what is the probability that both students have fewer than two absences? Express the answer to the nearest hundredth.

												Total
Absences	10	9	8	7	6	5	4	3	2	1	0	
Frequency	2	1	4	8	5	15	12	18	26	37	12	$\sum\limits_{i=0}^{10} f_i = 140$

Solution The frequencies tell us the number of students.

There are 140 students and 49 students have fewer than 2 absences.

$$n(S) = {}_{140}C_2 = \frac{140 \times 139}{2 \times 1} = 9{,}730$$

$$n(E) = {}_{49}C_2 = \frac{49 \times 48}{2 \times 1} = 1{,}176$$

The probability that both students have fewer than 2 absences $= \frac{1{,}176}{9{,}730} \approx .12$.

Answer .12

Geometric Probabilities

There are geometric applications to probability.

For example, the graph shows the region on the coordinate axis that is enclosed by the rectangle whose vertices have the coordinates $(-a, 0)$, $(a, 0)$, (a, a), and $(-a, a)$, and the graph shows the inequality $y \geq |x|$. If a point within the rectangle is chosen at random, what is the probability that the coordinates of the point make the inequality $y \geq |x|$ true? To answer this question we must assume that regions of equal area contain the same number of points.

Let S, the sample space, be the set of points in the rectangle.

Let E, the event, be the set of points in the rectangle that are a point of $y \geq |x|$.
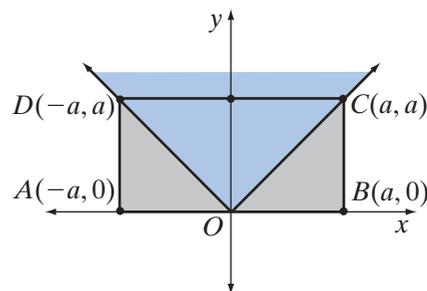
$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Area of } \triangle CDO}{\text{Area of rectangle } ABCD} = \frac{\frac{1}{2}a(2a)}{a(2a)} = \frac{1}{2}$$

EXAMPLE 3

A student wrote a computer program that will generate random pairs of numbers. The first element of each pair is between 0 and 4 and the second element is between 0 and 3. What is the theoretical probability that a pair of numbers generated by the program is in the interior of a circle whose equation is $(x - 1)^2 + (y - 1)^2 = 1$?

Solution
- Every pair of numbers generated by the computer program is in the interior of a rectangle whose length is 4 and whose width is 3. The area of this rectangle is 12.

- The center of the circle is the point $(1, 1)$ and the radius is 1.

- The area of this circle is $\pi(1)^2 = \pi$.
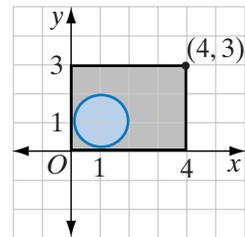
The interior of the circle is a subset of the interior of the rectangle as shown in the graph at the right.

$$P(\text{point is in the interior of the circle}) = \frac{\text{Area of the circle}}{\text{Area of the rectangle}} = \frac{\pi}{12} \quad \textit{Answer}$$

Exercises

Writing About Mathematics

1. A seed company tests 2,000 tomato seeds and obtains 1,954 plants. The seed company advertises that their seed is 97% productive.

   a. Is the seed company's claim justified?

   b. Is the seed company's claim based on theoretical or empirical probability?

2. Three students, Alex, Beth, and Casey, auditioned for the lead in the spring play. The director felt that the probability that Alex would be given the part was .35 and that the probability that Beth would be given the part was .25. Is it possible to determine the probability that Casey would be given the part? Justify your answer.

Developing Skills

3. What is the probability of getting a 2 on a single throw of a fair die?

4. What is the probability of getting a number greater than 2 on a single throw of a fair die?

5. What is the probability of getting a sum of 8 when a pair of dice are thrown?

6. The buyer of a lottery ticket chooses four numbers from the numbers 1 to 32. Repetition is not allowed.

    a. How many combinations of four numbers are possible?

    b. What is the probability of choosing all four of the winning numbers?

7. A standard deck of cards contains 52 cards divided into 4 suits. There are two red suits: hearts and diamonds, and two black suits: clubs and spades. Each suit contains 13 cards; ace, king, queen, jack, and cards numbered 2 through 10. A card is drawn from a standard deck without replacement. What is the probability that the card is a king?

8. Two cards are drawn from a standard deck of 52 cards without replacement. What is the probability that both cards are kings?

9. Three cards are drawn from a standard deck of 52 cards without replacement. What is the probability that all three cards are kings?

10. The letters of the word SEED are arranged at random. What is the probability that the arrangement begins and ends with E?

11. The letters of the word TOMATO are arranged at random. What is the probability that the arrangement begins and ends with T?

12. The letters of the word MATHEMATICS are arranged at random. What is the probability that the arrangement begins and ends with M?

13. The letters of the word STRAWBERRIES are written on cards and the cards are then shuffled. A card is picked at random. Find the probability that the card contains:

    a. the letter S   **b.** the letter R

    c. a consonant   **d.** a letter in the word CREAM

14. A piggy bank contains 60 pennies, 25 nickels, 10 dimes, and 5 quarters. If it is equally likely that any one of the coins will fall out when the bank is turned upside down, what is the probability that the coin is:

    a. a penny?   **b.** a dime?

   **c.** not a quarter?   **d.** a nickel or a dime?

Applying Skills

15. Three construction companies bid on a renovation project for a supermarket. The owner of Well-Built estimates that the competing companies have probabilities of .2 and .4, respectively, of getting the job. What is the probability that Well-Built will get the job?

16. A coin that was tossed 1,000 times came up heads 526 times. Determine the experimental probabilities for heads and tails.

17. A die was rolled 1,200 times and a 5 came up 429 times.

    a. Find the experimental probability for rolling a 5.

    b. Based on a comparison of the experimental and theoretical probabilities, do you think the die is fair? Explain your answer.

18. The weather report gives the probability of rain on Saturday as 40% and the probability of rain on Sunday as 60%. What is the probability that it will rain on Saturday but not rain on Sunday?

19. A variety box of instant oatmeal contains 10 plain, 6 maple, and 4 apple-cinnamon flavored packets. Ernestine reaches in and takes 3 packets without looking. Find each probability:

a. *P*(2 plain)   **b.** *P*(1 maple, 1 apple-cinnamon)

c. *P*(2 plain, 1 maple)   **d.** *P*(1 of each flavor)

20. The quality control department of a company that produces flashbulbs finds that 1 out of 1,000 bulbs tested fails to function properly. The flashbulbs are sold in packages of four. What is the probability that all the bulbs in a package will function properly?

21. There are 3 seniors and 15 juniors in Mrs. Gillis's math class. Three students are chosen at random from the class.

a. What is the probability that the group consists of a senior and two juniors?

b. If the group consists of a senior and two juniors, what is the probability that Stephanie, a senior, and Jan, a junior, are chosen?

22. Of the 18 students in Mrs. Shusda's math class, 12 take chemistry. If three students are absent from the class today, what is the probability that none of them take chemistry?

23. A play area for children has activities that are designed for taller and others for shorter children. The heights in inches of children at this play area last Saturday are given in the table.

Height	40	41	42	43	44	45	46	47	48	49	50
Frequency	1	1	2	5	9	14	8	6	3	0	1

a. For one of the activities, a child must be at least 42 inches tall. What is the probability that a group of four children who want participate in this activity are all tall enough?

b. For another activity, the child's height must be less than 45 inches. What is the probability that a group of five children who want to participate in this activity are all short enough?

24. At a carnival, Chrystal is managing a game in which a dart is thrown at a square board with a bull's-eye in the center. The board measures 3 feet by 3 feet and the bull's-eye has a 1-inch radius. Players who hit the bull's-eye receive a prize.

3 ft

3 ft

a. Assume that each player is unskilled and each throw is random but always lands within the square. What is the theoretical probability that a player will hit the bull's-eye?

b. At the end of the day, Chrystal finds that she gave out 72 prizes for 1,270 throws. What is the experimental probability that a player hit the bull's-eye?

c. Are the theoretical probability and the empirical probability the same? If not, explain why they are different.
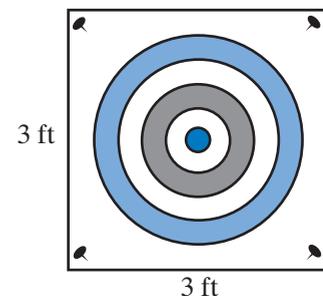
25. An irregular figure is drawn on a graph within a square that measures 6 inches on each side. The theoretical probability that the coordinates of a point that lies within the square also lies within the irregular figure is $\frac{4}{9}$. What is the area of the irregular figure?

26. A group of friends are playing a game in which each player chooses a number from 1 to 6 and takes a turn tossing a die until the chosen number appears on the die. The required number of tosses is the person's score for that round of play.

   a. What is the probability that a player gets a score of 3 on the first round?

   b. What is the probability that a player gets a score of 5 on the first round?

   c. What is the probability of a score of n?

   d. Show that the probabilities of scores that are consecutive integers from 1 to 5 form a geometric sequence.

27. A carton of 24 eggs contains 4 eggs with double yolks. If 3 eggs are selected at random, determine each probability to the nearest hundredth:

   a. All 3 eggs will have single yolks.

   b. 2 eggs will have single yolks and 1 egg will have a double yolk.

28. Inheritance of different genes can produce 6 different types of coats in cattle: straight red (SSRR or SsRR), curly red (ssRR), straight white (SSrr or Ssrr), curly white (ssrr), straight roan (SSRr or SsRr), or curly roan (ssRr).

Straight Roan Parent

		SR	Sr	sR	sr
Curly Roan Parent	sR	SsRR	SsRr	ssRR	ssRr
	sr	SsRr	Ssrr	ssRr	ssrr
	sR	SsRR	SsRr	ssRR	ssRr
	sr	SsRr	Ssrr	ssRr	ssrr

Use the Punnett square given above to determine the probability that one parent with a straight roan coat (SsRr) and one with a curly roan coat (ssRr) will produce an offspring with:

a. a straight roan coat   **b.** a curly roan coat

16-4 PROBABILITY WITH TWO OUTCOMES

In previous sections, we found the probability of a particular result on all trials. For example, if the probability that a team will win a game is $\frac{3}{4}$, then the probability of losing a game is $1 - \frac{3}{4}$ or $\frac{1}{4}$.

- $P(\text{win all of the first five games}) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{243}{1,024}$

- $P(\text{lose all of the first five games}) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{1,024}$

What is the probability that the team will win some games and lose other games? We will examine all possibilities.

CASE 1 *Win four out of five games*

There are five ways this can happen:

$\{W, W, W, W, L\}$ $P(\text{win all but the fifth}) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{81}{1,024}$

$\{W, W, W, L, W\}$ $P(\text{win all but the fourth}) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{81}{1,024}$

$\{W, W, L, W, W\}$ $P(\text{win all but the third}) = \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{1,024}$

$\{W, L, W, W, W\}$ $P(\text{win all but the second}) = \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{1,024}$

$\{L, W, W, W, W\}$ $P(\text{win all but the first}) = \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{1,024}$

The number of ways that the team can win four out of five games is $_5C_4 = \frac{5!}{4!} = 5$.

$$P(\text{win 4 out of 5 games}) = {}_5C_4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 = 5\left(\frac{81}{256}\right)\left(\frac{1}{4}\right) = \frac{405}{1,024}$$

We can use the same reasoning for each of the other possible combinations of wins and losses.

CASE 2 *Win three out of five games*

The team can win the first three games and lose the fourth and fifth:

$$P(\{W, W, W, L, L\}) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} = \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = \frac{27}{1,024}$$

There are other orders in which the team can win three and lose two. The number of combinations of three wins out of five games is $_5C_3 = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$. For each of these 10 possibilities, the probability is $\frac{27}{1,024}$.

$$P(\text{3 wins out of 5 games}) = {}_5C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = 10\left(\frac{27}{64}\right)\left(\frac{1}{16}\right) = \frac{270}{1,024}$$

CASE 3 *Win two out of five games*

The team can win the first two games and lose the third, fourth and fifth:

$$P(\{W, W, L, L, L\}) = \tfrac{3}{4} \times \tfrac{3}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} = \left(\tfrac{3}{4}\right)^2 \left(\tfrac{1}{4}\right)^3 = \tfrac{9}{1,024}$$

There are other orders in which the team can win two and lose three. The number of combinations of two wins out of five games is $_5C_2 = \tfrac{5 \times 4}{2 \times 1} = 10$. For each of these 10 possibilities, the probability is $\tfrac{9}{1,024}$.

$$P(2 \text{ wins out of 5 games}) = {}_5C_2 \left(\tfrac{3}{4}\right)^2 \left(\tfrac{1}{4}\right)^3 = 10\left(\tfrac{9}{16}\right)\left(\tfrac{1}{64}\right) = \tfrac{90}{1,024}$$

CASE 4 *Win one out of five games*

The team can win the first game and lose the last four:

$$P(\{W, L, L, L, L\}) = \tfrac{3}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} = \left(\tfrac{3}{4}\right)^1 \left(\tfrac{1}{4}\right)^4 = \tfrac{3}{1,024}$$

There are other orders in which the team can win one and lose four. The number of combinations of one win out of five games is $_5C_1 = \tfrac{5}{1} = 5$. For each of these 5 possibilities, the probability is $\tfrac{3}{1,024}$.

$$P(1 \text{ win out of 5 games}) = {}_5C_1 \left(\tfrac{3}{4}\right)^1 \left(\tfrac{1}{4}\right)^4 = 5\left(\tfrac{3}{4}\right)\left(\tfrac{1}{256}\right) = \tfrac{15}{1,024}$$

We have found the probability that the team will win 5, 4, 3, 2, 1, or 0 games. The sum of the probabilities must be 1. *Check:*

$$P(5 \text{ wins}) + P(4 \text{ wins}) + P(3 \text{ wins}) + P(2 \text{ wins}) + P(1 \text{ win}) + P(0 \text{ wins}) = 1$$

$$\tfrac{243}{1,024} + \tfrac{405}{1,024} + \tfrac{270}{1,024} + \tfrac{90}{1,024} + \tfrac{15}{1,024} + \tfrac{1}{1,024} = \tfrac{1,024}{1,024} = 1 ✔$$

The example given above illustrates the solution to finding exactly r successes in n independent trials in a **Bernoulli experiment** or **binomial experiment**. In a Bernoulli experiment, the n trials are independent and each trial has only two outcomes: success and failure. For instance, if a die is tossed, and success is showing 1, then failure is not showing 1. If a card is drawn from a deck of 52 cards, the trials are independent only if the card drawn is replaced after each draw. If success is drawing a king, then failure is not drawing a king.

In general, for a Bernoulli experiment, if the probability of success is p and the probability of failure is $1 - p = q$, then the probability of r successes in n independent trials is:

$$_nC_r \, p^r q^{n-r}$$

We refer to this formula as the **binomial probability formula**.

EXAMPLE 1

A waiter knows from experience that 7 out of 10 people who dine alone will leave a tip. Tuesday evening, the waiter served 12 lone diners.

a. Is this a Bernoulli experiment?

b. What is the probability, to the nearest thousandth, that the waiter received a tip from 9 of these diners?

Solution **a.** There are two outcomes to this experiment: the diner may leave a tip or the diner may not leave a tip. For each diner, leaving a tip or not leaving a tip is an independent outcome. Therefore, this is a Bernoulli experiment. *Answer*

b. If 7 out of 10 lone diners leave a tip, the probability that a lone diner will leave a tip is $\frac{7}{10}$ or .7 and the probability that a lone diner will not leave a tip is $1 - .7 = .3$.

$$P(9 \text{ tips from 12 diners}) = {}_{12}C_9 \,(.7)^9 \,(.3)^3 \approx .240 \ \textit{Answer}$$

EXAMPLE 2

What is the probability that 2 shows on three of the dice when four dice are tossed?

Solution These are independent trials with $n = 4$ and $r = 3$. The number shown on any die is independent of that shown on any other.

$$P(2) = p = \tfrac{1}{6}$$
$$P(\text{not } 2) = q = 1 - \tfrac{1}{6} = \tfrac{5}{6}$$
$$P(2 \text{ exactly three out of four times}) = {}_4C_3 \, p^3 q^1$$
$$= \tfrac{4 \times 3 \times 2}{3 \times 2 \times 1}\left(\tfrac{1}{6}\right)^3\left(\tfrac{5}{6}\right)^1$$
$$= 4\left(\tfrac{1}{216}\right)\left(\tfrac{5}{6}\right) = \tfrac{20}{1,296} \ \textit{Answer}$$

EXAMPLE 3

A company that makes breakfast cereal puts a coupon for a free box of cereal in 3 out of every 20 boxes. What is the probability that Mrs. Sullivan will find 2 coupons in the next 5 boxes of cereal that she buys?

Solution $\quad P(\text{box contains a coupon}) = \tfrac{3}{20}$

$$P(\text{box does not contain a coupon}) = 1 - \tfrac{3}{20} = \tfrac{17}{20}$$
$$P(2 \text{ coupons in 5 boxes}) = {}_5C_2 \, p^2 q^3$$
$$= \tfrac{5 \times 4}{2 \times 1}\left(\tfrac{3}{20}\right)^2\left(\tfrac{17}{20}\right)^3$$
$$= 10\left(\tfrac{9}{400}\right)\left(\tfrac{4,913}{8,000}\right) = \tfrac{442,170}{3,200,000} \ \textit{Answer}$$

Binomial Probabilities and the Graphing Calculator

We can use the calculator to find a binomial probability with the binompdf(function of the DISTR menu. To use the binompdf(function, we must supply the number of trials n, the probability of success p, and the number of successes r:

$$\text{binompdf}(n, p, r)$$

For example, to find the probability of winning 2 out of 3 games $\left(\text{where the probability of winning is } \frac{3}{4}\right)$, enter:

ENTER: **2nd**   **DISTR**

Scroll to binompdf(and press ENTER.

ENTER: 3 **,** 3 **÷** 4 **,** 2 **)** **ENTER**

DISPLAY:
```
BINOMPDF(3,3/4,2
)
              .421875
```

The probability of winning 2 out of 3 games is approximately .4219.

To calculate the probabilities of winning 0 to 3 games, use the cumulative probability function, binomcdf(. To use the binomcdf(function, we must supply the number of trials n, the probability of success p, and the number of successes r:

$$\text{binomcdf}(n, p, r)$$

If r is not supplied, a list of probabilities for 0 to n trials is returned.

ENTER: **2nd**   **DISTR**

Scroll to binomcdf(and press ENTER.

ENTER: 3 **,** 3 **÷** 4 **)** **ENTER**

DISPLAY:
```
BINOMCDF(3,3/4)
 {.015625   .15625...
```

The calculator returns the set {.015625, .15625, .578125, 1}.

$P(0 \text{ games out of } 3) \approx .015625$ $P(0 \text{ or } 1 \text{ games out of } 3) \approx .15625$

$P(0, 1, \text{ or } 2 \text{ games out of } 3) \approx .578125$ $P(0, 1, 2, \text{ or } 3 \text{ games out of } 3) \approx 1$

We can use these cumulative probabilities to find the individual probabilities.

$P(0 \text{ wins}) \approx .015625$

$P(1 \text{ win}) \approx .15625 - .015625 = .140625$

$P(2 \text{ wins}) \approx .578125 - .15625 = .421875$

$P(3 \text{ wins}) \approx 1 - .578125 = .421875$

Exercises

Writing About Mathematics

1. There are 20 students in a club, 12 boys and 8 girls. If five members of the club are chosen at random to represent the club at a competition, what is the probability that in the group chosen there are exactly 2 boys? Explain why this is not a Bernoulli experiment.

2. Hunter said that the number of combinations of n things taken r at a time is equal to the number of permutations of n things taken n at a time when r are identical; that is, $_nC_r = \frac{n!}{r!}$. Do you agree with Hunter? Explain why or why not.

Developing Skills

In 3–6, find exact probabilities showing all required computation.

3. A fair coin is tossed five times. What is the probability that the coin lands heads:

    a. exactly once?   **b.** exactly twice?   **c.** exactly three times?

    d. exactly four times?   **e.** exactly five times?   **f.** zero times?

    g. Which is the most likely event(s) when tossing a coin five times?

4. A fair die is tossed five times. What is the probability of tossing a 6:

    a. exactly once?   **b.** exactly twice?   **c.** exactly three times?

    d. exactly four times?   **e.** exactly five times?   **f.** zero times?

    g. Which is the most likely event(s) when tossing a die five times?

5. A card is drawn and replaced four times from a standard deck of 52 cards. What is the probability of drawing a king: **a.** exactly once?   **b.** exactly twice?

    c. exactly three times?   **d.** exactly four times?   **e.** zero times?

    f. Which is the most likely event(s) when drawing a card four times with replacement?

6. A multiple-choice test of 10 questions has 4 choices for each question. Only one choice is correct. A student who did not study for the test guesses at each answer.

    a. What is the probability that that student will have exactly 5 correct answers?

    b. What is the probability that that student will have only 1 correct answer?

Applying Skills

In 7–14, answers can be rounded to four decimal places.

7. The probability that our team will win a basketball game is $\frac{2}{3}$. What is the probability that they will win exactly 5 of the next 7 games?

8. Jack's batting average is .2. What is the probability that he will get one hit in the next three times at bat?

9. Marie plays solitaire on her computer and keeps a record of her wins and losses. Her record shows that she won 950 of the last 1,000 games that she played. What is the probability that she will win three of the next four games that she plays?

10. A fast-food restaurant gives coupons for 10% off of the next purchase with 1 out of every 5 purchases.

  a. What is the probability that Zoe will receive a coupon with her next purchase?

  b. What is the probability that Zoe will receive just one coupon with her next three purchases?

11. A store estimates that 1 out of every 25 customers is returning or exchanging merchandise. In the last hour, a cashier had 5 customers. What is the probability that exactly 2 of those were making returns or exchanges?

12. Assume that there are an equal number of births in each month so that the probability is $\frac{1}{12}$ that a person chosen at random was born in May. A group of 20 friends meet monthly and celebrate the birthdays for that month. What is the probability that at the May celebration, exactly two members of the group have May birthdays?

13. Using a special type of fishing hook, researchers discovered that the hooking mortality rate was approximately 8%. That is, of the fish caught and released with this hook, about 8% eventually died. A fisherman caught and released 5 fish using this hook. What is the probability that all 5 fish will live?

14. Statisticians investigating the Internet-surfing habits of students at a large high school discovered that only 35% are interested in protecting their privacy online. What is the probability that none of four randomly chosen students are interested in protecting their privacy?

16-5 BINOMIAL PROBABILITY AND THE NORMAL CURVE

At Least and At Most

When we are anticipating success or failure in future events, we are often interested in success in at least r of the next n trials or failure in at most r of the next n trials.

What is the probability of getting at least 8 heads when a coin is tossed 10 times? We will get at least 8 heads if we land 8, 9, or 10 heads.

$$P(\text{at least 8 heads}) = P(8 \text{ heads}) + P(9 \text{ heads}) + P(10 \text{ heads})$$

$$= {}_{10}C_8\left(\tfrac{1}{2}\right)^8\left(\tfrac{1}{2}\right)^2 + {}_{10}C_9\left(\tfrac{1}{2}\right)^9\left(\tfrac{1}{2}\right)^1 + {}_{10}C_{10}\left(\tfrac{1}{2}\right)^{10}\left(\tfrac{1}{2}\right)^0$$

$$= 45\left(\tfrac{1}{1,024}\right) + 10\left(\tfrac{1}{1,024}\right) + 1\left(\tfrac{1}{1,024}\right)$$

$$= \tfrac{56}{1,024} = .0546875$$

In this example, we are asked to compute the probability of landing at least 8 heads or at least 8 successes. The table below shows common English expressions that are equivalent to "at least 8 successes."

Expression	Binomial Probability
At least 8 successes 8 or more successes No fewer than 8 successes	$r \geq 8$, $P\left(\begin{smallmatrix}\text{at least}\\ 8 \text{ successes}\end{smallmatrix}\right) = P(8 \text{ successes}) + P(9 \text{ successes}) + \cdots + P(n \text{ successes})$
At most 8 successes 8 or fewer successes No more than 8 successes	$r \leq 8$, $P\left(\begin{smallmatrix}\text{at most}\\ 8 \text{ successes}\end{smallmatrix}\right) = P(0 \text{ successes}) + P(1 \text{ success}) + \cdots + P(8 \text{ successes})$
More than 8 successes Exceeding 8 successes	$r > 8$, $P\left(\begin{smallmatrix}\text{more than}\\ 8 \text{ successes}\end{smallmatrix}\right) = P(9 \text{ successes}) + P(10 \text{ successes}) + \cdots + P(n \text{ successes})$
Fewer than 8 successes Under 8 successes	$r < 8$, $P\left(\begin{smallmatrix}\text{fewer than}\\ 8 \text{ successes}\end{smallmatrix}\right) = P(0 \text{ successes}) + P(1 \text{ success}) + \cdots + P(7 \text{ successes})$

EXAMPLE 1

Find the probability that when nine dice are cast, 5 will show on at most seven dice.

Solution The probability of success on one die is $\frac{1}{6}$.

The probability of failure on one die is $\frac{5}{6}$.

Success means that 5 shows on 0, 1, 2, 3, 4, 5, 6, or 7 dice.

Failure means that 5 shows on 8 or 9 dice.

Probability of success and of failure can be written in sigma notation.

$$P(\text{success}) = \sum_{r=0}^{7} {}_9C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{9-r} \qquad P(\text{failure}) = \sum_{r=8}^{9} {}_9C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{9-r}$$

Since there are fewer cases for the probability of failure, we will evaluate $P(\text{failure})$ and use the fact that $P(\text{success}) + P(\text{failure}) = 1$.

$$P(\text{failure}) = \sum_{r=8}^{9} {}_9C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{9-r} = {}_9C_8\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^1 + {}_9C_9\left(\frac{1}{6}\right)^9\left(\frac{5}{6}\right)^0$$

$$= 9\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^1 + 1\left(\frac{1}{6}\right)^9\left(\frac{5}{6}\right)^0$$

$$= \frac{45}{10{,}077{,}696} + \frac{1}{10{,}077{,}696}$$

$$= \frac{46}{10{,}077{,}696} \approx 0.0000045645$$

$$P(\text{success}) = 1 - P(\text{failure}) = 1 - \frac{46}{10{,}077{,}696}$$

$$= \frac{10{,}077{,}650}{10{,}077{,}696}$$

$$\approx 0.9999954355 \;\; \textit{Answer}$$

Area Under a Normal Curve

In Section 15-6, we learned about the normal distribution, a special type of distribution that frequently occurs in nature. Recall that in a normal distribution:

- Approximately 68% of the data falls within one standard deviation of the mean.

- Approximately 95% of the data falls within two standard deviations of the mean.

- Approximately 99.7% of the data falls within three standard deviations of the mean.
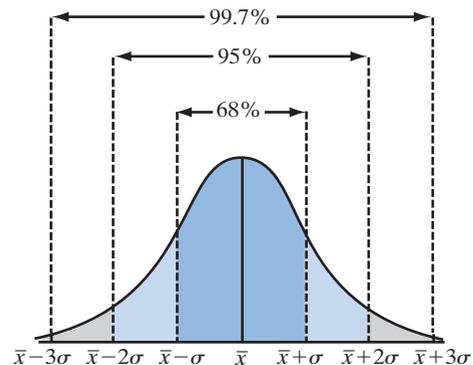
The area under a normal curve can be interpreted as a probability. The area can be found by using the normalcdf(function of the graphing calculator. The normalcdf(function requires the following inputs:

$$\text{normalcdf}(low, high, mean, standard\ devation)$$

This will find the area or probability under the normal curve from *low* to *high* with the given *mean* and *standard deviation*.

EXAMPLE 2

The height of a certain variety of plant is normally distributed with a mean of 50 centimeters and a standard deviation of 15 centimeters. What is the probability that the height of a randomly selected plant is between 40 and 55 centimeters?

Solution   mean = 50   standard deviation = 15

ENTER: [2nd] [DISTR] [2] 40 [,] 55 [,] 50 [,] 15 [)] [ENTER]

DISPLAY:

```
NORMALCDF(40,55,
50, 15)
         .3780661293
```

The probability that the height of a randomly selected plant is between 40 and 55 centimeters is approximately .378. *Answer*

The Normal Approximation to the Binomial Distribution

When 10 coins are tossed, the outcome can show 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 heads. The table gives the number of possible combination of coins on which heads occur for each number of heads and the probability of that number of heads.

No. of Heads	No. of Combinations	Probability
0	1	.00098
1	10	.00977
2	45	.04395
3	120	.11719
4	210	.20508
5	252	.24609
6	210	.20508
7	120	.11719
8	45	.04395
9	10	.00977
10	1	.00098
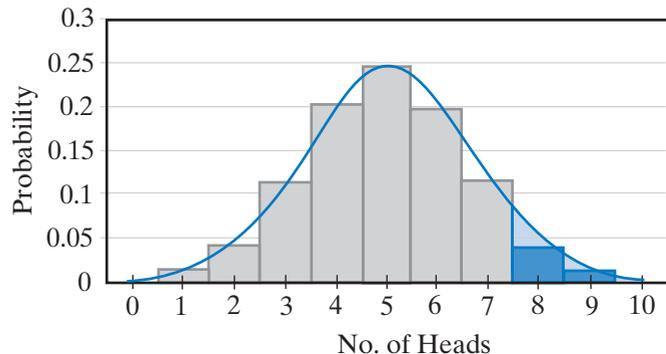
The graph above shows the probability of landing each number of heads. When we draw a smooth curve through these points, the figure is a bell-shaped curve that approximates a normal curve. Geometrically, each bar has width 1 and each height is equal to the corresponding probability at that value. Thus, the probability of getting at least 8 heads is equal to the sum of the *areas* of the bars

labeled 8, 9, and 10. Alternatively, since the distribution can be approximated by a normal curve, the probability of getting at least 8 heads is approximately equal to the area under the normal curve as shown in the graph at the bottom of page 703. However, before we can use the normal distribution in this way, we first need to find the mean and standard deviation of this distribution.

It can be shown that for a **binomial distribution** of n trials with a probability of success of p, the mean is

$$np$$

and the standard deviation is

$$\sqrt{np(1 - p)}$$

For this distribution

$$np = 10\left(\tfrac{1}{2}\right) = 5 \qquad \sqrt{np(1 - p)} = \sqrt{10\left(\tfrac{1}{2}\right)\left(1 - \tfrac{1}{2}\right)} \approx 1.58$$

The normal distribution with mean 5 and standard deviation 1.58 is said to make a good **normal approximation to the binomial distribution**. The probability of getting at least 8 heads when a coin is tossed 10 times is approximately equal to the area under this normal curve from 7.5 (the boundary value between 7 and 8) to 10.5 (the boundary value for the largest value). The area under the curve can be found by using the normalcdf(function of the graphing calculator:

ENTER: [2nd] [DISTR] [2] 7.5 [,] 10.5 [,] 5 [,] 1.58 [ )] [ENTER]

DISPLAY:

```
NORMALCDF(7.5, 10
.5, 5, 1.58)
          .0565432019
```
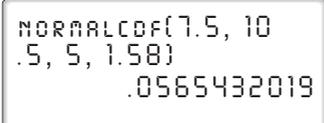
The calculator gives the probability as .0565432019. Compare this value with the exact value of .0546875 found at the beginning of the section. As the number of trials increases, the normal approximation more closely approximates the exact value.

Why should we use the normal approximation? Suppose, for example, that we want to find the probability of getting at least 60 heads in 100 tosses of a fair coin. Here any number of heads greater than or equal to 60 is a success. There are 41 possible results:

$$P(\text{at least 60 heads}) = \sum_{r=60}^{100} {}_{100}C_r\left(\tfrac{1}{2}\right)^r\left(\tfrac{1}{2}\right)^{100-r}$$

We can obtain an approximate answer by considering the area under the normal curve. The mean is $np = 100(.5)$ or 50 and the standard deviation is $\sqrt{np(1 - p)} = \sqrt{100(.5)(1 - .5)}$ or 5. A calculator can be used to evaluate the answer.

ENTER: [2nd]   [DISTR]   [2]   59.5   [,]   100.5   [,]   50   [,]   5   [)]   [ENTER]

DISPLAY:

```
NORMALCDF(59.5, 1
00.5, 50, 5)
          .0287164928
```
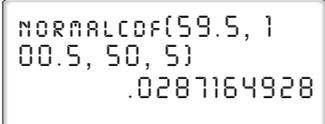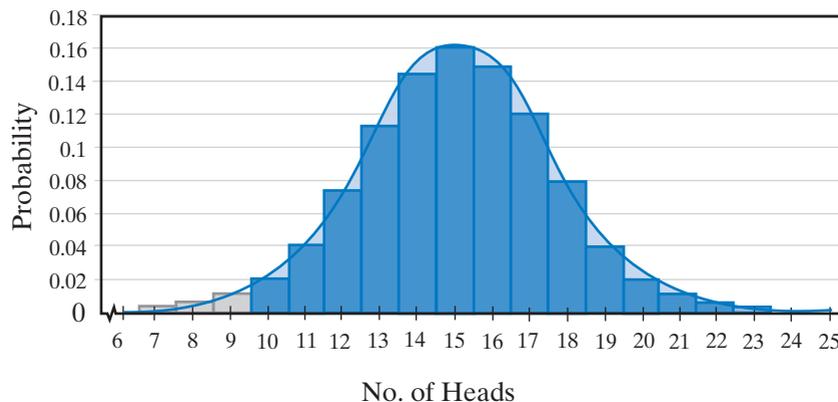
The probability of getting at least 60 heads in 100 tosses of a fair coin is approximately equal to .0287164928.

EXAMPLE 3

The probability that a team will win a game is $\frac{3}{5}$. Use the normal approximation to estimate the probability that the team will win at least 10 of its next 25 games.

Solution

No. of Heads

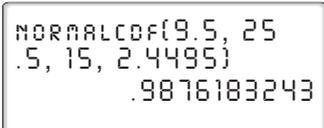$p = $ probability of success $= \frac{3}{5}$ $q = $ probability of failure $= \frac{2}{5}$

$n = 25$ $r = 10$

The mean and standard deviation of this binomial distribution are

$$np = (25)\left(\tfrac{3}{5}\right) = 15 \qquad \sqrt{npq} = \sqrt{25\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)} \approx 2.4495$$

The probability of winning at least 10 of the next 25 games is approximately equal to the area under the normal curve from 9.5 to 25.5.

ENTER: [2nd] [DISTR] [2] 9.5 [,] 25.5 [,] 15 [,] 2.4495 [)]   [ENTER]

DISPLAY:
```
NORMALCDF(9.5, 25
.5, 15, 2.4495)
          .9876183243
```

Answer .988

Note: When using the normal approximation, the standard deviation needs to be rounded to at least one more decimal place than the desired probability.

Exercises

Writing About Mathematics

1. Emma said that if the probability of success in at least r out of n trials is z, then the probability of success in at most r out of n trials is $1 - z$. Do you agree with Emma? Explain why or why not.

2. If the probability of success for one trial is .1, does the area under the normal curve give a good approximation of the probability of success in at least 8 out of 10 trials? Explain why or why not. (*Hint:* Consider the shape of the curve.)

Developing Skills

In 3–5, find exact probabilities showing all required computation.

3. Five fair coins are tossed. Find the probability of the coins showing:

  a. at least four heads   **b.** at least three heads

  c. at least two heads   **d.** at least one head

4. Three fair dice are tossed. Find the probability of the dice showing:

  a. at most one 6   **b.** at most two 6's

  c. at least two 6's   **d.** at least one 6

5. A spinner is divided into five equal sections numbered 1 through 5. The arrow is equally likely to land on any section. Find the probability of:

  a. an odd number on any one spin

  b. at least three odd numbers on four spins

  c. at least two odd numbers on four spins

  d. at least one odd number on four spins

In 6–9, write an expression using sigma notation that can be used to find each probability.

6. At least 10 heads when 15 coins are tossed

7. At most 7 tails when 10 coins are tossed

8. At least 5 wins in the next 20 games when $P(\text{win}) = \frac{2}{3}$

9. At most 3 losses in the next 10 games when $P(\text{win}) = \frac{2}{3}$

In 10–13, the mean and standard deviation of a normal distribution are given. Find each probability to the nearest hundredth.

10. mean $= 80$, standard deviation $= 10$, $P(50 \le x \le 95)$

11. mean $= 50$, standard deviation $= 2.5$, $P(44 \le x \le 50)$

12. mean $= 6$, standard deviation $= 3$, $P(2 \le x \le 7)$

13. mean $= 8$, standard deviation $= 1$, $P(7 \le x)$

In 14–17, use the normal approximation to estimate each probability. Round your answers to three decimal places.

14. $P(\text{at least 20 successes})$, $p = \frac{1}{6}$, $n = 100$ **15.** $P(\text{more than 20 successes})$, $p = \frac{1}{10}$, $n = 200$

16. $P(\text{fewer than 60 successes})$, $p = \frac{5}{6}$, $n = 75$ **17.** $P(\text{at most 100 successes})$, $p = \frac{6}{7}$, $n = 125$

Applying Skills

18. A family with 5 children is selected at random. What is the probability that the family has at least 3 boys?

19. A quality control department tests a product and finds that the probability of a defective part is .001. What is the probability that in a run of 100 items, no more than 2 are defective?

20. A landscape company will replace any shrub that they plant if it fails to grow. They estimate that the probability of failure to grow is .02. What is the probability that, of the 200 shrubs planted this week, at most 3 must be replaced?

21. When adjusted for inflation, the monthly amount that a historic restaurant spent on cleaning for a 30-year period was normally distributed with a mean of $2,100 and a standard deviation of $240.

  a. What is the probability that the restaurant will spend between $2,500 and $2,800 on cleaning for the month of January?

  b. If the restaurant has only $2,400 allotted for cleaning for the month of January, what is the probability that it will exceed its budget for cleaning?

In 22–24, use the normal approximation to estimate each probability. Round your answers to three decimal places.

22. A fair coin is tossed 100 times. What is the probability of at most 48 heads?

23. A fair die is tossed 70 times. What is the probability of at last 40 even numbers?

24. The probability that a basketball team will win any game is .5. What is the probability that the team will win at least 12 out of 25 games?

16-6 THE BINOMIAL THEOREM

The algebraic expressions $5a + 3$, $a^2 - 2a$, $3p + 2q$, and $b + c$ are all binomials. Any binomial can be expressed as $x + y$ with x equal to the first term and y equal to the second term. When n is a positive integer, $(x + y)^n$ can be expressed as a polynomial called the **binomial expansion**.

$$(x + y)^1 = 1x + 1y$$
$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

A pattern of powers of x and y develop as we write the binomial expansions. Consider the last expansion shown above: $(x + y)^5$.

Write $(x + y)^5$ as a product of five factors:

$$(x + y)(x + y)(x + y)(x + y)(x + y)$$

This product is the sum of all possible combinations of an x or a y from each term.

- We can choose the y from zero factors and x from all five factors to obtain the term in x^5. There is $_5C_0$ or 1 way to do this, so the term is $1x^5$.
- We can choose y from one factor and x from four factors to obtain the term in x^4y. There are $_5C_1$ or 5 ways to do this, so the term is $5x^4y$.
- We can choose y from two factors and x from three factors to obtain the term in x^3y^2. There are $_5C_2$ or 10 ways to do this, so the term is $10x^3y^2$.
- We can choose y from three factors and x from two factors to obtain the term in x^2y^3. There are $_5C_3$ or 10 ways to do this, so the term is $10x^2y^3$.
- We can choose y from four factors and x from one factor to obtain the term in x^1y^4. There are $_5C_4$ or 5 ways to do this, so the term is $5xy^4$.
- We can choose y from all five factors in one way to obtain the term in y^5. There is $_5C_5$ or 1 way to do this, so the term is $1y^5$.

Compare the coefficients of the terms of a binomial expansion with the combinations $_nC_r$. Include $(x + y)^0 = 1$ in this comparison.

$$
\begin{array}{lll}
(x + y)^0 = & 1 & 1 \\
(x + y)^1 = & {}_1C_0 x + {}_1C_1 y & 1\ 1 \\
(x + y)^2 = & {}_2C_0 x^2 + {}_2C_1 xy + {}_2C_2 y^2 & 1\ 2\ 1 \\
(x + y)^3 = & {}_3C_0 x^3 + {}_3C_1 x^2y + {}_3C_2 x^1y^2 + {}_3C_3 y^3 & 1\ 3\ 3\ 1 \\
(x + y)^4 = & {}_4C_0 x^4 + {}_4C_1 x^3y + {}_4C_2 x^2y^2 + {}_4C_3 x^1y^3 + {}_4C_4 y^4 & 1\ 4\ 6\ 4\ 1 \\
(x + y)^5 = & {}_5C_0 x^5 + {}_5C_1 x^4y + {}_5C_2 x^3y^2 + {}_5C_3 x^2y^3 + {}_5C_4 x^1y^4 + {}_5C_5 y^5 & 1\ 5\ 10\ 10\ 5\ 1
\end{array}
$$

The pattern shown at the right above is called **Pascal's Triangle**. Each row of Pascal's Triangle has one more entry than the row above. Each row begins and ends with 1. Each entry between the first and last entry are the sum of the two entries from the row above. For example, in the fourth row:

$$
\begin{array}{ccccc}
 & 1 & 2 & 1 & \\
1 & 3 & & 3 & 1
\end{array}
$$

The next row of Pascal's Triangle would be:

$$
\begin{array}{ccccccc}
 & 1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}
$$

$$
{}_6C_0 \quad {}_6C_1 \quad {}_6C_2 \quad {}_6C_3 \quad {}_6C_4 \quad {}_6C_5 \quad {}_6C_6
$$

We can write the general expression for the expansion of a binomial:

$$(x + y)^n = \sum_{i=0}^{n} {}_nC_i x^{n-i}y^i$$

Note that in the expansion of $(x + y)^n$, there are $n + 1$ terms and the rth term is:

$${}_nC_{r-1}\, x^{n-r+1}\, y^{r-1}$$

EXAMPLE I

Write the expansion of $(x + y)^8$.

Solution
$$
\begin{aligned}
(x + y)^8 &= \sum_{i=0}^{8} {}_8C_i x^{8-i}y^i \\
&= {}_8C_0 x^8 + {}_8C_1 x^7y + {}_8C_2 x^6y^2 + {}_8C_3 x^5y^3 + {}_8C_4 x^4y^4 + {}_8C_5 x^3y^5 \\
&\quad + {}_8C_6 x^2y^6 + {}_8C_7 xy^7 + {}_8C_8 y^8 \\
&= 1x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 \\
&\quad + 8xy^7 + 1y^8 \ \textit{Answer}
\end{aligned}
$$

EXAMPLE 2

Write the expansion of $(3a - 2)^4$

Solution

$$(x + y)^4 = {}_4C_0\, x^4 + {}_4C_1\, x^3y + {}_4C_2\, x^2y^2 + {}_4C_3\, x^1y^3 + {}_4C_4\, y^4$$
$$= 1x^4 + 4x^3y + 6x^2y^2 + 4x^1y^3 + 1y^4$$

Let $x = 3a$ and $y = -2$.

$$(3a - 2)^4 = 1(3a)^4 + 4(3a)^3(-2) + 6(3a)^2(-2)^2 + 4(3a)(-2)^3 + (-2)^4$$
$$= 81a^4 - 216a^3 + 216a^2 - 96a + 16 \quad \textit{Answer}$$

EXAMPLE 3

Write the 4th term of the expansion of $(b + 5)^{10}$.

Solution   Let $n = 10$, $r = 4$, $x = b$, and $y = 5$.

METHOD 1	**METHOD 2**
The 4th term of $(x + y)^{10}$ is the term that chooses $(10 - 4 + 1)$ or 7 factors of x and $(4 - 1)$ or 3 factors of y:	Use the formula:

METHOD 1

$${}_{10}C_3 x^7 y^3$$
$$= {}_{10}C_3 (b)^7(5)^3$$
$$= 120b^7(125)$$
$$= 15{,}000b^7$$

METHOD 2

The 4th term $= {}_nC_{r-1}\, x^{n-r+1}y^{r-1}$
$$= {}_{10}C_{4-1}\, (b)^{10-4+1}(5)^{4-1}$$
$$= {}_{10}C_3\, b^7 5^3$$
$$= 120b^7(125)$$
$$= 15{,}000b^7$$

Answer $15{,}000b^7$

Exercises

Writing About Mathematics

1. Explain why the expansion of $(x + y)^n = \sum_{i=0}^{n} {}_nC_i x^{n-i}y^i$ can also be written as

$(x + y)^n = \sum_{i=0}^{n} {}_nC_{n-i} x^{n-i}y^i$.

2. Ariel said that $\left(x + \frac{1}{x}\right)^n$ is equal to $\sum_{i=0}^{n} {}_nC_i x^{n-2i}$. Do you agree with Ariel? Explain why or why not.

Developing Skills

In 3–10, write the expansion of each binomial.

3. $(x + y)^6$ **4.** $(x + y)^7$ **5.** $(1 + y)^5$ **6.** $(x + 2)^5$

7. $(a + 3)^4$ **8.** $(2 + a)^4$ **9.** $(2b - 1)^3$ **10.** $(1 - i)^5; i = \sqrt{-1}$

11. Write 10 lines of the Pascal Triangle, starting with 1.

In 12–17, write the nth term of each binomial expansion.

12. $(x + y)^{15}, n = 3$ **13.** $(x + y)^{10}, n = 7$ **14.** $(2x + y)^6, n = 4$

15. $(x - y)^9, n = 5$ **16.** $(3a + 2b)^7, n = 6$ **17.** $\left(y - \frac{1}{y}\right)^8, n = 4$

In 18–20, for the given expansion, identify which term is shown and write the next term.

18. $(a - 2b)^8; 1{,}792a^3b^5$ **19.** $(5c - 2d)^4; -160cd^3$ **20.** $\left(\frac{x}{2} - 2y\right)^{11}; \frac{165}{4}x^7y^4$

Applying Skills

21. The length of a side of a cube is represented by $(3x - 1)$. Use the binomial theorem to write a polynomial that represents the volume of the cube.

22. When $100 are invested at 4% interest compounded quarterly, the value of the investment after 3 years is $100(1 + 0.01)^{12}$. Use the binomial theorem to express the value in sigma notation.

23. A machine purchased for $75,000 is expected to decrease in value by 20% each year. The value of the machine after n years is $75{,}000(1 - 0.20)^n$. Use the binomial theorem to express the value of the machine after 5 years in sigma notation.

CHAPTER SUMMARY

The set of all possible **outcomes** for an activity is called the **sample space**. An **event** is a subset of the sample space.

The **Counting Principle**: If one activity can occur in any of m ways and a second activity can occur in any of n ways, then both activities can occur in the order given in $m \cdot n$ ways.

A **permutation** is an arrangement of objects in a specific order. The number of permutations of n things taken r at a time, $r \leq n$, is written $_nP_r$.

$$_nP_r = n(n - 1)(n - 2) \ldots (n - r + 1) = \frac{n!}{(n - r)!}$$

The number of permutations of n things taken n at a time when a are identical is $\frac{n!}{a!}$.

A **combination** is a selection of objects in which order is not important. The number of combinations of n things taken r at a time is symbolized $_nC_r$ or as $\binom{n}{r}$.

$$_nC_r = \frac{_nP_r}{_rP_r} \text{ or } \binom{n}{r} = \frac{_nP_r}{r!}.$$

Probability is an estimate of the frequency of an outcome in repeated trials. **Theoretical probability** is the ratio of the number of elements in an event to the number of equally likely elements in the sample space. **Empirical probability** or **experimental probability** is the best estimate, based on a large number of trials, of the ratio of the number of successful results to the number trials.

Let $P(E)$ represent the probability of an event E.

$n(E)$ be the number of element in event E.

$n(S)$ be the number of elements in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

The probability form of the **Counting Principle**: If E_1 and E_2 are independent events, and if $P(E_1) = p$ and $P(E_2) = q$, then the probability of both E_1 and E_2 occurring is $p \times q$.

An event can be any subset of the sample space, including the empty set and the sample space itself: $0 \le n(E) \le n(S)$. Therefore since $n(S)$ is a positive number, $\frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(S)}{n(S)}$ or $0 \le P(E) \le 1$. $P(E) = 0$ when the event is the empty set, that is, there are no outcomes. $P(E) = 1$ when the event is the sample space, that is, certainty. $P(E) + P(\text{not } E) = 1$.

A **Bernoulli experiment** or **binomial experiment** is an experiment with two independent outcomes: success or failure. In general, for a Bernoulli experiment, if the probability of success is p and the probability of failure is $1 - p = q$, then the probability of r successes in n independent trials is:

$$_nC_r p^r q^{n-r}$$

- The probability of at most r successes in n trials is $\sum\limits_{i=0}^{r} {_nC_i} p^i q^{n-i}$.

- The probability of at least r successes in n trials is $\sum\limits_{i=r}^{n} {_nC_i} p^i q^{n-i}$.

The area under a normal curve between two x-values is equal to the probability that a randomly selected x-value is between the two given values.

For a binomial distribution of n trials with a probability of success of p, the mean is np and the standard deviation is $\sqrt{np(1 - p)}$. A binomial distribution can be approximated by a normal distribution with this given mean and standard deviation.

When n is a positive integer, $(x + y)^n$ can be expressed as a polynomial called the **binomial expansion**:

$$(x + y)^n = \sum_{i=0}^{n} {}_nC_i x^{n-i} y^i$$

In the expansion of $(x + y)^n$, there are $n + 1$ terms and the rth term is:

$${}_nC_{r-1} x^{n-r+1} y^{r-1}$$

VOCABULARY

16-1 Outcome • Sample space • Counting Principle • Tree diagram • Event • Dependent events • Independent events

16-2 Permutation • Combination

16-3 Probability • Theoretical probability • Empirical probability • Experimental probability

16-4 Bernoulli experiment • Binomial experiment • Binomial probability formula

16-5 Binomial distribution • Normal approximation to the binomial distribution

16-6 Binomial expansion • Pascal's Triangle

REVIEW EXERCISES

In 1–12, evaluate each expression.

1. $6!$

2. ${}_5P_5$

3. ${}_5C_5$

4. ${}_8P_6$

5. ${}_8C_6$

6. ${}_8C_0$

7. ${}_8C_8$

8. $\binom{10}{6}$

9. $\binom{50}{49}$

10. $\frac{12!}{10!}$

11. $\frac{12!}{5!\,7!}$

12. ${}_{40}C_{39}$

13. In how many different ways can 8 students choose a seat if there are 8 seats in the classroom?

14. In how many different ways can 8 students choose a seat if there are 12 seats in the classroom?

15. Randy is trying to arrange the letters T, R, O, E, E, M to form distinct arrangements. How many possible arrangements are there?

16. A buffet offers six flavors of ice cream, four different syrups, and five toppings. How many different sundaes consisting of ice cream, syrup, and topping are possible?

17. A dish contains eight different candies. How many different choices of three can be made?

18. At a carnival, the probability that a person will win a prize at the ring-toss game has been found to be $\frac{1}{20}$. What is the probability that a prize will be won by exactly

   a. 1 of the next 5 players?   **b.** 3 of the next 5 players?

   c. 0 of the next 5 players?   **d.** 2 of the next 20 players?

19. A fair coin is tossed 12 times. What is the probability of the coin showing

   a. exactly 10 heads?   **b.** at least 10 heads?   **c.** at most 10 heads?

20. A convenience store makes over \$200 approximately 75% of the days it is open. Use the normal approximation to determine the probability that the store makes over \$200 at least 265 out of 365 days. Round your answer to the nearest thousandth.

21. The math portion of the SAT is designed so that the scores are normally distributed with a mean of 500 and a standard deviation of 100. The maximum possible score is 800. To the nearest thousandth, what is the probability that a randomly selected student's score on the math section is

   a. more than 700?   **b.** less than 300?   **c.** between 400 and 600?

In 22–25, write the expansion of each binomial.

22. $(x + y)^4$ **23.** $(2a + 1)^7$ **24.** $(3 - x)^6$ **25.** $\left(b - \frac{1}{b}\right)^8$

26. When $i = \sqrt{-1}$, express $(1 + i)^5$ in $a + bi$ form.

27. Write in simplest form the fourth term of $(x - y)^9$.

28. Write in simplest form the seventh term of $(a + 3)^{10}$.

29. Write in simplest form the middle term of $(2x - 1)^{12}$.

Exploration

The winner of the World Series is determined by a *seven-game series*, where two teams play against each other until one team wins four games. Let p represent the probability that the American League team will win a game in the World Series.

a. Explain why the probability that a team will win the World Series is *not*

$$_7C_4\, p^4(1-p)^3.$$

b. Write an expression for the probability that the American League team will win the World Series in exactly 4 games.

c. Write an expression for the probability that the American League team will win the World Series in exactly 5 games. (*Hint:* In order for the series to run 5 games, the winning team must win 3 out of the first 4 games and then win the last game.)

d. Write an expression for the probability that the American League team will win the World series in exactly 6 games. (*Hint:* In order for the series to run 6 games, the winning team must win 3 out of the first 5 games and then win the last game.)

e. Write an expression for the probability that the American League team will win the World Series in exactly 7 games.

f. Write an expression for the probability that the American League team will win the World Series.

CUMULATIVE REVIEW   CHAPTERS 1–16

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. In simplest form, $(3x - 9) \div \left[12\left(x - \frac{9}{x}\right)\right]$ is equal to

(1) $\frac{5}{4x}$

(3) $\frac{1}{4(x+3)}$

(2) $\frac{x}{4x+3}$

(4) $\frac{x}{4(x+3)}$

2. Which of the following sets of ordered pairs is *not* a function?

(1) $\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$

(2) $\{(1,2),(2,2),(3,2),(4,2),(5,2)\}$

(3) $\{(1,2),(1,4),(1,6),(1,8),(1,10)\}$

(4) $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$

3. The sum of the roots of a quadratic equation is -2 and the product is -6. Which of the following could be the equation?

(1) $x^2 - 2x + 6 = 0$

(3) $x^2 + 2x + 6 = 0$

(2) $x^2 - 2x - 6 = 0$

(4) $x^2 + 2x - 6 = 0$

4. If $f(x) = x^2$ and $g(x) = 2x - 1$, then $f(g(-2))$ is

(1) 25   (2) 15   (3) 9   (4) 7

5. The product $(\sqrt{-4})(\sqrt{-12})$ is equal to

(1) $4\sqrt{3}$ (2) $\sqrt{48}$ (3) $4i\sqrt{3}$ (4) $-4\sqrt{3}$

6. Which of the following is an arithmetic sequence?

(1) $3, 1, -1, -3, \ldots$ (3) $3, -1, \frac{1}{3}, -\frac{1}{3}, \ldots$

(2) $3, -1, 1, -3, \ldots$ (4) $1, 1, 2, 3, 5, 8, \ldots$

7. Which of the following is equivalent to $\log a + \log b = 2 \log c$?

(1) $\log (a + b) = \log 2c$

(2) $\log (a + b) = \log c^2$

(3) $\log ab = \log 2c$

(4) $\log ab = \log c^2$

8. Which of the following is an identity?

(1) $\sec^2 \theta - 1 = \tan^2 \theta$

(2) $\tan^2 \theta - 1 = \sec^2 \theta$

(3) $\tan^2 \theta - \cot^2 \theta = 1$

(4) $\cos \theta = \sqrt{1 + \sin^2\theta}$

9. Which is not an equation of the graph shown at the right?

(1) $y = \sin \left(x - \frac{\pi}{2} \right)$

(2) $y = -\cos x$

(3) $y = \sin \left(x + \frac{\pi}{2} \right)$

(4) $y = \cos (x + \pi)$

10. The scores of a standardized test have a normal distribution with a mean of 75 and a standard deviation of 6. Which interval contains 95% of the scores?

(1) 57–93   (2) 63–87   (3) 69–81   (4) 57–87
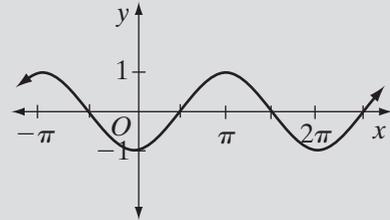
Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. If $\log_2 \frac{1}{4} = \log_8 x$, what is the value of x?

12. Find the solution set of the equation $\sqrt{3x + 1} = 2x$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. a. Draw the graph of $f(x) = (1.6)^x$ from $x = -1$ to $x = 4$.

    b. Write an expression for $f^{-1}(x)$ and draw the graph of f^{-1} for $0 < x \leq 5$.

14. The length of a rectangle is 3 feet less than twice the width. The area of the rectangle is less than 20 square feet. Find the possible widths of the rectangle.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. An environmental group organized a campaign to clean up debris in the city. Each Saturday, volunteer crews worked at the project. The number of volunteers increased each week as shown in the following record.

Week	1	2	3	4	5	6	7	8	9
Volunteers	30	50	67	78	85	87	91	94	97

    a. Draw a scatter plot to display the data.

    b. Write the non-linear equation that best represents this data.

    c. What would be the expected number of volunteers on the 10th Saturday?

16. Find, to the nearest degree, all of the values of θ in the interval $0° \leq \theta \leq 360°$ that satisfy the equation $2 \cos^2 \theta + 2 \sin \theta - 1 = 0$.