CHAPTER

13

TRIGONOMETRIC EQUATIONS

The triangle is a rigid figure, that is, its shape cannot be changed without changing the lengths of its sides. This fact makes the triangle a basic shape in construction. The theorems of geometry give us relationships among the measures of the sides and angles of triangle. Precisely calibrated instruments enable surveyors to obtain needed measurements. The identities and formulas of trigonometry enable architects and builders to formulate plans needed to construct the roads, bridges, and buildings that are an essential part of modern life.

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13-1 FIRST-DEGREE TRIGONOMETRIC EQUATIONS

A **trigonometric equation** is an equation whose variable is expressed in terms of a trigonometric function value. To solve a trigonometric equation, we use the same procedures that we used to solve algebraic equations. For example, in the equation $4 \sin \theta + 5 = 7$, $\sin \theta$ is multiplied by 4 and then 5 is added. Thus, to solve for $\sin \theta$, first add the opposite of 5 and then divide by 4.

$$\begin{aligned} 4 \sin \theta + 5 &= 7 \\ -5 &= -5 \\ \hline 4 \sin \theta &= 2 \\ \frac{4 \sin \theta}{4} &= \frac{2}{4} \\ \sin \theta &= \frac{1}{2} \end{aligned}$$

We know that $\sin 30° = \frac{1}{2}$, so one value of θ is 30°; $\theta_1 = 30$. We also know that since $\sin \theta$ is positive in the second quadrant, there is a second-quadrant angle, θ_2, whose sine is $\frac{1}{2}$. Recall the relationship between an angle in any quadrant to the acute angle called the reference angle. The following table compares the degree measures of θ from $-90°$ to $360°$, the radian measures of θ from $-\frac{\pi}{2}$ to 2π, and the measure of its reference angle.

	Fourth Quadrant	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
Angle	$-90° < \theta < 0°$ $-\frac{\pi}{2} < \theta < 0$	$0° < \theta < 90°$ $0 < \theta < \frac{\pi}{2}$	$90° < \theta < 180°$ $\frac{\pi}{2} < \theta < \pi$	$180° < \theta < 270°$ $\pi < \theta < \frac{3\pi}{2}$	$270° < \theta < 360°$ $\frac{3\pi}{2} < \theta < 2\pi$
Reference Angle	$-\theta$ $-\theta$	θ θ	$180° - \theta$ $\pi - \theta$	$\theta - 180°$ $\theta - \pi$	$360° - \theta$ $2\pi - \theta$

The reference angle for the second-quadrant angle whose sine is $\frac{1}{2}$ has a degree measure of 30° and $\theta_2 = 180° - 30°$ or 150°. Therefore, $\sin \theta_2 = \frac{1}{2}$. For $0° \leq \theta < 360°$, the solution set of $4 \sin \theta + 5 = 7$ is $\{30°, 150°\}$. In radian measure, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

In the example given above, it was possible to give the exact value of θ that makes the equation true. Often it is necessary to use a calculator to find an approximate value. Consider the solution of the following equation.

$$\begin{aligned} 5 \cos \theta + 7 &= 3 \\ 5 \cos \theta &= -4 \\ \cos \theta &= -\frac{4}{5} \\ \theta &= \arccos\left(-\frac{4}{5}\right) \end{aligned}$$

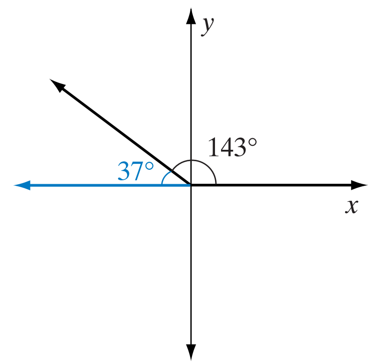
When we use a calculator to find θ, the calculator will return the value of the function $y = \arccos x$ whose domain is $0^\circ \leq x \leq 180^\circ$ in degree measure or $0 \leq x \leq \pi$ in radian measure.

In degree measure:

ENTER: 2nd COS^{-1} (-) 4 ÷ 5) ENTER

DISPLAY:
```
cos⁻¹(-4/5)
        143.1301024
```

To the nearest degree, one value of θ is 143°. In addition to this second-quadrant angle, there is a third-quadrant angle such that $\cos \theta = -\frac{4}{5}$. To find this third-quadrant angle, find the reference angle for θ.



Let R be the measure of the reference angle of the second-quadrant angle. That is, R is the acute angle such that $\cos \theta = -\cos R$.

$$R = 180^\circ - \theta = 180^\circ - 143^\circ = 37^\circ$$

The measure of the third-quadrant angle is:

$$\theta = R + 180^\circ$$
$$\theta = 37^\circ + 180^\circ$$
$$\theta = 217^\circ$$

For $0^\circ \leq \theta \leq 360^\circ$, the solution set of $5 \cos \theta + 7 = 3$ is $\{143^\circ, 217^\circ\}$. If the value of θ can be any angle measure, then for all integral values of n, $\theta = 143 + 360n$ or $\theta = 217 + 360n$.

Procedure

To solve a linear trigonometric equation:

1. Solve the equation for the function value of the variable.
2. Use a calculator or your knowledge of the exact function values to write one value of the variable to an acceptable degree of accuracy.
3. If the measure of the angle found in step 2 is not that of a quadrantal angle, find the measure of its reference angle.
4. Use the measure of the reference angle to find the degree measures of each solution in the interval $0^\circ \leq \theta < 360^\circ$ or the radian measures of each solution in the interval $0 \leq \theta < 2\pi$.
5. Add $360n$ (n an integer) to the solutions in degrees found in steps 2 and 4 to write all possible solutions in degrees. Add $2\pi n$ (n an integer) to the solutions in radians found in steps 2 and 4 to write all possible solutions in radians.

The following table will help you find the locations of the angles that satisfy trigonometric equations. The values in the table follow from the definitions of the trigonometric functions on the unit circle.

	Sign of a and b ($0 < \lvert a \rvert < 1, b \neq 0$)	
	+	−
$\sin\theta = a$	Quadrants I and II	Quadrants III and IV
$\cos\theta = a$	Quadrants I and IV	Quadrants II and III
$\tan\theta = b$	Quadrants I and III	Quadrants II and IV

EXAMPLE 1

Find the solution set of the equation $7\tan\theta = 2\sqrt{3} + \tan\theta$ in the interval $0° \leq \theta < 360°$.

Solution *How to Proceed*

(1) Solve the equation for $\tan\theta$:

$$7\tan\theta = 2\sqrt{3} + \tan\theta$$
$$6\tan\theta = 2\sqrt{3}$$
$$\tan\theta = \frac{\sqrt{3}}{3}$$

(2) Since $\tan\theta$ is positive, θ_1 can be a first-quadrant angle:

$$\theta_1 = 30°$$

(3) Since θ is a first-quadrant angle, $R = \theta$:

$$R = 30°$$

(4) Tangent is also positive in the third quadrant. Therefore, there is a third-quadrant angle such that $\tan\theta = \frac{\sqrt{3}}{3}$. In the third quadrant, $\theta_2 = 180° + R$:

$$\theta_2 = 180° + R$$
$$\theta_2 = 180° + 30° = 210°$$

Answer The solution set is $\{30°, 210°\}$.

EXAMPLE 2

Find, to the nearest hundredth, all possible solutions of the following equation in radians:

$$3(\sin A + 2) = 3 - \sin A$$

Solution

How to Proceed

(1) Solve the equation for $\sin A$:

$$3(\sin A + 2) = 3 - \sin A$$
$$3 \sin A + 6 = 3 - \sin A$$
$$4 \sin A = -3$$
$$\sin A = -\tfrac{3}{4}$$

(2) Use a calculator to find one value of A (be sure that the calculator is in RADIAN mode):

ENTER: 2nd SIN⁻¹ (-) 3 ÷ 4) ENTER

DISPLAY:

```
sin⁻¹(⁻3/4)
        ⁻.848062079
```

One value of A is -0.848.

(3) Find the reference angle: $R = -A = -(-0.848) = 0.848$

(4) Sine is negative in quadrants III and IV. Use the reference angle to find a value of A in each of these quadrants:

In quadrant III: $A_1 = \pi + 0.848 \approx 3.99$
In quadrant IV: $A_2 = 2\pi - 0.848 \approx 5.44$

(5) Write the solution set: $\{3.99 + 2\pi n, 5.44 + 2\pi n\}$ *Answer*

Note: When $\sin^{-1}\left(-\frac{3}{4}\right)$ was entered, the calculator returned the value in the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, the range of the inverse of the sine function. This is the measure of a fourth-quadrant angle that is a solution of the equation. However, solutions are usually given as angle measures in radians between 0 and 2π plus multiples of 2π. Note that the value returned by the calculator is $5.43 + 2\pi(-1) \approx -0.85$.

EXAMPLE 3

Find all possible solutions to the following equation in degrees:

$$\tfrac{1}{2}(\sec \theta + 3) = \sec \theta + \tfrac{5}{2}$$

Solution (1) Solve the equation for $\sec\theta$:

$$\tfrac{1}{2}(\sec\theta + 3) = \sec\theta + \tfrac{5}{2}$$
$$\sec\theta + 3 = 2\sec\theta + 5$$
$$-2 = \sec\theta$$

(2) Rewrite the equation in terms of $\cos\theta$: $\cos\theta = -\frac{1}{2}$

(3) Use a calculator to find one value of θ:

ENTER: **2nd** **COS⁻¹** -1 **÷** 2 **)** **ENTER**

DISPLAY:
```
cos⁻¹(-1/2)
                120
```

Cosine is negative in quadrant II so $\theta_1 = 120°$.

(4) Find the reference angle: $R = 180 - 120 = 60°$

(5) Cosine is also negative in quadrant III. Use the reference angle to find a value of θ_2 in quadrant III: In quadrant III: $\theta_2 = 180 + 60 = 240°$

(6) Write the solution set: $\{120 + 360n, 240 + 360n\}$ *Answer*

Trigonometric Equations and the Graphing Calculator

Just as we used the graphing calculator to approximate the irrational solutions of quadratic-linear systems in Chapter 5, we can use the graphing calculator to approximate the irrational solutions of trigonometric equations. For instance,

$$3(\sin A + 2) = 3 - \sin A$$

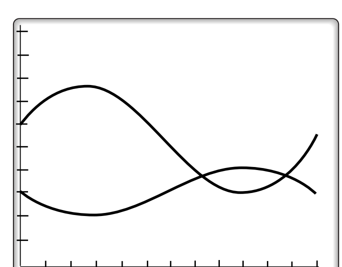
from Example 2 can be solved using the intersect feature of the calculator.

STEP 1. Treat each side of the equation as a function.
Enter $Y_1 = 3(\sin X + 2)$ and $Y_2 = 3 - \sin X$ into the **Y=** menu.

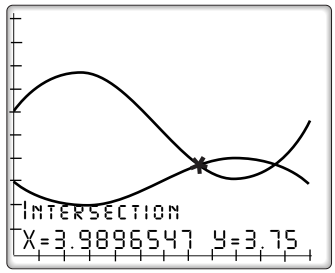
STEP 2. Using the following viewing window:

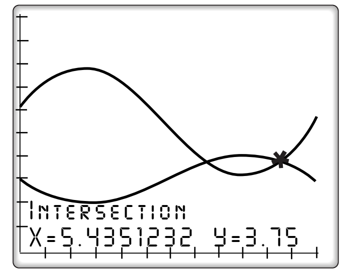
$$Xmin = 0, Xmax = 2\pi, Xscl = \frac{\pi}{6},$$
$$Ymin = 0, Ymax = 10$$

and with the calculator set to radian mode, **GRAPH** the functions.



STEP 3. The solutions are the x-coordinates of the intersection points of the graphs. We can find the intersection points by using the intersect function. Press **2nd** **CALC** 5 **ENTER** **ENTER** to select both curves. When the calculator asks you for a guess, move the cursor near one of the intersection points using the arrow keys and then press **ENTER**. Repeat this process to find the other intersection point.





As before, the solutions in the interval $0 \leq \theta < 2\pi$ are approximately 3.99 and 5.44. The solution set is $\{3.99 + 2\pi n, 5.44 + 2\pi n\}$.

Exercises

Writing About Mathematics

1. Explain why the solution set of the equation $2x + 4 = 8$ is $\{2\}$ but the solution set of the equation $2 \sin x + 4 = 8$ is $\{\ \}$, the empty set.

2. Explain why $2x + 4 = 8$ has only one solution in the set of real numbers but the equation $2 \tan x + 4 = 8$ has infinitely many solutions in the set of real numbers.

Developing Skills

In 3–8, find the exact solution set of each equation if $0° \leq \theta < 360°$.

3. $2 \cos \theta - 1 = 0$

4. $3 \tan \theta + \sqrt{3} = 0$

5. $4 \sin \theta - 1 = 2 \sin \theta + 1$

6. $5(\cos \theta + 1) = 5$

7. $3(\tan \theta - 2) = 2 \tan \theta - 7$

8. $\sec \theta + \sqrt{2} = 2\sqrt{2}$

In 9–14, find the exact values for θ in the interval $0 \leq \theta < 2\pi$.

9. $3 \sin \theta - \sqrt{3} = \sin \theta$

10. $5 \cos \theta + 3 = 3 \cos \theta + 5$

11. $\tan \theta + 12 = 2 \tan \theta + 11$

12. $\sin \theta + \sqrt{2} = \frac{\sqrt{2}}{2}$

13. $3 \csc \theta + 5 = \csc \theta + 9$

14. $4(\cot \theta + 1) = 2(\cot \theta + 2)$

In 15–20, find, to the nearest degree, the measure of an acute angle for which the given equation is true.

15. $\sin\theta + 3 = 5\sin\theta$

16. $3\tan\theta - 1 = \tan\theta + 9$

17. $5\cos\theta + 1 = 8\cos\theta$

18. $4(\sin\theta + 1) = 6 - \sin\theta$

19. $\csc\theta - 1 = 3\csc\theta - 11$

20. $\cot\theta + 8 = 3\cot\theta + 2$

In 21–24, find, to the nearest tenth, the degree measures of all θ in the interval $0° \leq \theta < 360°$ that make the equation true.

21. $8\cos\theta = 3 - 4\cos\theta$

22. $5\sin\theta - 1 = 1 - 2\sin\theta$

23. $\tan\theta - 4 = 3\tan\theta + 4$

24. $2 - \sec\theta = 5 + \sec\theta$

In 25–28, find, to the nearest hundredth, the radian measures of all θ in the interval $0 \leq \theta < 2\pi$ that make the equation true.

25. $10\sin\theta + 1 = 3 - 2\sin\theta$

26. $9 - 2\cos\theta = 8 - 4\cos\theta$

27. $15\tan\theta - 7 = 5\tan\theta - 3$

28. $\cot\theta - 6 = 2\cot\theta + 2$

Applying Skills

29. The voltage E (in volts) in an electrical circuit is given by the function

$$E = 20\cos(\pi t)$$

where t is time in seconds.

a. Graph the voltage E in the interval $0 \leq t \leq 2$.

b. What is the voltage of the electrical circuit when $t = 1$?

c. How many times does the voltage equal 12 volts in the first two seconds?

d. Find, to the nearest hundredth of a second, the times in the first two seconds when the voltage is equal to 12 volts.

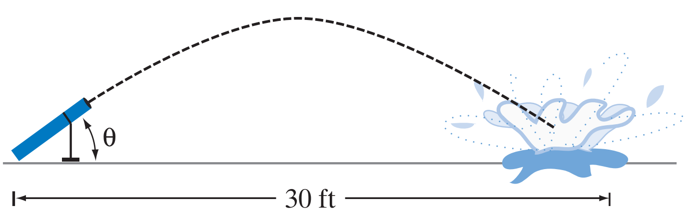
(1) Let $\theta = \pi t$. Solve the equation $20\cos\theta = 12$ in the interval $0 \leq \theta < 2\pi$.

(2) Use the formula $\theta = \pi t$ and your answers to part (1) to find t when $0 \leq \theta < 2\pi$ and the voltage is equal to 12 volts.

30. A water balloon leaves the air cannon at an angle of θ with the ground and an initial velocity of 40 feet per second. The water balloon lands 30 feet from the cannon. The distance d traveled by the water balloon is given by the formula

$$d = \frac{1}{32}v^2 \sin 2\theta$$

where v is the initial velocity.



a. Let $x = 2\theta$. Solve the equation $30 = \frac{1}{32}(40)^2 \sin\ x$ to the nearest tenth of a degree.

b. Use the formula $x = 2\theta$ and your answer to part **a** to find the measure of the angle that the cannon makes with the ground.

31. It is important to understand the underlying mathematics before using the calculator to solve trigonometric equations. For example, Adrian tried to use the intersect feature of his graphing calculator to find the solutions of the equation $\cot\theta = \sin\left(\theta - \frac{\pi}{2}\right)$ in the interval $0 \leq \theta \leq \pi$ but got an error message. Follow the steps that Adrian used to solve the equation:

(1) Enter $Y_1 = \frac{1}{\tan\ X}$ and $Y_2 = \sin\left(X - \frac{\pi}{2}\right)$ into the Y= menu.

(2) Use the following viewing window to graph the equations:

$$Xmin = 0, Xmax = \pi, Xscl = \tfrac{\pi}{6}, Ymin = -5, Ymax = 5$$

(3) The curves seem to intersect at $\left(\frac{\pi}{2}, 0\right)$. Press 2nd CALC 5 ENTER ENTER to select both curves. When the calculator asks for a guess, move the cursor near the intersection point using the arrow keys and then press ENTER.

a. Why does the calculator return an error message?

b. Is $\theta = \frac{\pi}{2}$ a solution to the equation? Explain.

13-2 USING FACTORING TO SOLVE TRIGONOMETRIC EQUATIONS

We know that the equation $3x^2 - 5x - 2 = 0$ can be solved by factoring the left side and setting each factor equal to 0. The equation $3\tan^2\theta - 5\tan\theta - 2 = 0$ can be solved for $\tan\theta$ in a similar way.

$$\begin{array}{r|l} \multicolumn{2}{c}{3x^2 - 5x - 2 = 0} \\ \multicolumn{2}{c}{(3x + 1)(x - 2) = 0} \\ 3x + 1 = 0 & x - 2 = 0 \\ 3x = -1 & x = 2 \\ x = -\frac{1}{3} & \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{c}{3\tan^2\theta - 5\tan\theta - 2 = 0} \\ \multicolumn{2}{c}{(3\tan\theta + 1)(\tan\theta - 2) = 0} \\ 3\tan\theta + 1 = 0 & \tan\theta - 2 = 0 \\ 3\tan\theta = -1 & \tan\theta = 2 \\ \tan\theta = -\frac{1}{3} & \end{array}$$

In the solution of the algebraic equation, the solution is complete. The solution set is $\left\{-\frac{1}{3}, 2\right\}$. In the solution of the trigonometric equation, we must now find the values of θ.

There are two values of θ in the interval $0° \leq \theta < 360°$ for which $\tan\theta = 2$, one in the first quadrant and one in the third quadrant. The calculator will display the measure of the first-quadrant angle, which is also the reference angle for the third-quadrant angle.

ENTER: **2nd** **TAN⁻¹** 2 **)** **ENTER**

DISPLAY:
```
TAN⁻¹(2)
       63.43494882
```

To the nearest tenth of a degree, the measure of θ in the first quadrant is 63.4. This is also the reference angle for the third-quadrant angle.

$$\begin{aligned} \text{In quadrant I:} &\quad \theta_1 = 63.4° \\ \text{In quadrant III:} &\quad \theta_2 = 180 + R = 180 + 63.4 = 243.4° \end{aligned}$$

There are two values of θ in the interval $0° \leq \theta < 360°$ for which $\tan\theta = -\frac{1}{3}$, one in the second quadrant and one in the fourth quadrant.

ENTER: **2nd** **TAN⁻¹** -1 **÷** 3 **)** **ENTER**

DISPLAY:
```
TAN⁻¹(⁻1/3)
      ⁻18.43494882
```

The calculator will display the measure of a fourth-quadrant angle, which is negative. To the nearest tenth of a degree, one measure of θ in the fourth quadrant is -18.4. The opposite of this measure, 18.4, is the measure of the reference angle for the second- and fourth-quadrant angles.

$$\begin{aligned} \text{In quadrant II:} &\quad \theta_3 = 180 - 18.4 = 161.6° \\ \text{In quadrant IV:} &\quad \theta_4 = 360 - 18.4 = 341.6° \end{aligned}$$

The solution set of $3\tan^2\theta - 5\tan\theta - 2 = 0$ is

$$\{63.4°, 161.6°, 243.4°, 341.6°\}$$

when $0° \leq \theta < 360°$.

EXAMPLE 1

Find all values of θ in the interval $0 \leq \theta < 2\pi$ for which $2\sin\theta - 1 = \frac{3}{\sin\theta}$.

Solution *How to Proceed*

(1) Multiply both sides of the equation by $\sin\theta$:

$$2\sin\theta - 1 = \frac{3}{\sin\theta}$$
$$2\sin^2\theta - \sin\theta = 3$$

(2) Write an equivalent equation with 0 as the right side:

$$2\sin^2\theta - \sin\theta - 3 = 0$$

(3) Factor the left side:

$$(2\sin\theta - 3)(\sin\theta + 1) = 0$$

(4) Set each factor equal to 0 and solve for $\sin\theta$:

$$\begin{array}{r|l} 2\sin\theta - 3 = 0 & \sin\theta + 1 = 0 \\ 2\sin\theta = 3 & \sin\theta = -1 \\ \sin\theta = \frac{3}{2} & \end{array}$$

(5) Find all possible values of θ: There is no value of θ such that $\sin\theta > 1$. For $\sin\theta = -1$, $\theta = \frac{3\pi}{2}$.

Answer $\theta = \frac{3\pi}{2}$

EXAMPLE 2

Find the solution set of $4\sin^2 A - 1 = 0$ for the degree measures of A in the interval $0° \le A < 360°$.

Solution

METHOD 1

Factor the left side.

$$4\sin^2 A - 1 = 0$$
$$(2\sin A - 1)(2\sin A + 1) = 0$$

$2\sin A - 1 = 0$	$2\sin A + 1 = 0$
$2\sin A = 1$	$2\sin A = -1$
$\sin A = \frac{1}{2}$	$\sin A = -\frac{1}{2}$

METHOD 2

Solve for $\sin^2 A$ and take the square root of each side of the equation.

$$4\sin^2 A - 1 = 0$$
$$4\sin^2 A = 1$$
$$\sin^2 A = \tfrac{1}{4}$$
$$\sin A = \pm\tfrac{1}{2}$$

If $\sin A = \frac{1}{2}$, $A = 30°$ or $A = 150°$. If $\sin A = -\frac{1}{2}$, $A = 210°$ or $A = 330°$.

Answer $\{30°, 150°, 210°, 330°\}$

Note: The graphing calculator does not use the notation $\sin^2 A$, so we must enter the square of the trig function as $(\sin A)^2$ or enter $\sin (A)^2$. For example, to check the solution $A = 30°$ for Example 2:

ENTER: 4 (SIN 30)) x^2 − 1 ENTER

4 SIN 30) x^2 − 1 ENTER

DISPLAY:

```
4(sin(30))²-1
                0
4sin(30)²-1
                0
```

Factoring Equations with Two Trigonometric Functions

To solve an equation such as $2\sin\theta\cos\theta + \sin\theta = 0$, it is convenient to rewrite the left side so that we can solve the equation with just one trigonometric function value. In this equation, we can rewrite the left side as the product of two factors. Each factor contains one function.

$$2 \sin \theta \cos \theta + \sin \theta = 0$$
$$\sin \theta\,(2 \cos \theta + 1) = 0$$

$$\begin{array}{r|l}
\sin \theta = 0 & 2 \cos \theta + 1 = 0 \\
\theta_1 = 0^\circ & \quad 2 \cos \theta = -1 \\
\theta_2 = 180^\circ & \quad \cos \theta = -\frac{1}{2} \\
 & \text{Since } \cos 60^\circ = \frac{1}{2}, R = 60^\circ. \\
 & \text{In quadrant II: } \theta_3 = 180 - 60 = 120^\circ \\
 & \text{In quadrant III: } \theta_4 = 180 + 60 = 240^\circ
\end{array}$$

The solution set is $\{0^\circ, 120^\circ, 180^\circ, 240^\circ\}$.

EXAMPLE 3

Find, in radians, all values of θ in the interval $0 \le \theta \le 2\pi$ that are in the solution set of:

$$\sec \theta \csc \theta + \sqrt{2} \csc \theta = 0$$

Solution Factor the left side of the equation and set each factor equal to 0.

$$\sec \theta \csc \theta + \sqrt{2} \csc \theta = 0$$
$$\csc \theta \left(\sec \theta + \sqrt{2}\right) = 0$$

$$\begin{array}{l|l}
\csc \theta = 0 \;✘ & \sec \theta + \sqrt{2} = 0 \\
\text{No solution} & \quad \sec \theta = -\sqrt{2} \\
 & \quad \cos \theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\
 & \text{Since } \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}, R = \frac{\pi}{4}. \\
 & \text{In quadrant II: } \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \\
 & \text{In quadrant III: } \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}
\end{array}$$

Answer $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$

Exercises

Writing About Mathematics

1. Can the equation $\tan \theta + \sin \theta \tan \theta = 1$ be solved by factoring the left side of the equation? Explain why or why not.

2. Can the equation $2(\sin \theta)(\cos \theta) + \sin \theta + 2 \cos \theta + 1 = 0$ be solved by factoring the left side of the equation? Explain why or why not.

Developing Skills

In 3–8, find the exact solution set of each equation if $0° \leq \theta < 360°$.

3. $2\sin^2\theta + \sin\theta - 1 = 0$

4. $3\tan^2\theta = 1$

5. $\tan^2\theta - 3 = 0$

6. $2\sin^2\theta - 1 = 0$

7. $6\cos^2\theta + 5\cos\theta - 4 = 0$

8. $2\sin\theta\cos\theta + \cos\theta = 0$

In 9–14, find, to the nearest tenth of a degree, the values of θ in the interval $0° \leq \theta < 360°$ that satisfy each equation.

9. $\tan^2\theta - 3\tan\theta + 2 = 0$

10. $3\cos^2\theta - 4\cos\theta + 1 = 0$

11. $9\sin^2\theta - 9\sin\theta + 2 = 0$

12. $25\cos^2\theta - 4 = 0$

13. $\tan^2\theta + 4\tan\theta - 12 = 0$

14. $\sec^2\theta - 7\sec\theta + 12 = 0$

In 15–20, find, to the nearest hundredth of a radian, the values of θ in the interval $0 \leq \theta < 2\pi$ that satisfy the equation.

15. $\tan^2\theta - 5\tan\theta + 6 = 0$

16. $4\cos^2\theta - 3\cos\theta = 1$

17. $5\sin^2\theta + 2\sin\theta = 0$

18. $3\sin^2\theta + 7\sin\theta + 2 = 0$

19. $\csc^2\theta - 6\csc\theta + 8 = 0$

20. $2\cot^2\theta - 13\cot\theta + 6 = 0$

21. Find the smallest positive value of θ such that $4\sin^2\theta - 1 = 0$.

22. Find, to the nearest hundredth of a radian, the value of θ such that $\sec\theta = \frac{5}{\sec\theta}$ and $\frac{\pi}{2} < \theta < \pi$.

23. Find two values of A such that $(\sin A)(\csc A) = -\sin A$.

13-3 USING THE QUADRATIC FORMULA TO SOLVE TRIGONOMETRIC EQUATIONS

Not all quadratic equations can be solved by factoring. It is often useful or necessary to use the quadratic formula to solve a second-degree trigonometric equation.

The trigonometric equation $2\cos^2\theta - 4\cos\theta + 1 = 0$ is similar in form to the algebraic equation $2x^2 - 4x + 1 = 0$. Both are quadratic equations that cannot be solved by factoring over the set of integers but can be solved by using the quadratic formula with $a = 2$, $b = -4$, and $c = 1$.

Algebraic equation:	*Trigonometric equation*:
$2x^2 - 4x + 1 = 0$	$2\cos^2\theta - 4\cos\theta + 1 = 0$
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$\cos\theta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)}$	$\cos\theta = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)}$
$x = \dfrac{4 \pm \sqrt{16 - 8}}{4}$	$\cos\theta = \dfrac{4 \pm \sqrt{16 - 8}}{4}$
$x = \dfrac{4 \pm \sqrt{8}}{4}$	$\cos\theta = \dfrac{4 \pm \sqrt{8}}{4}$
$x = \dfrac{2 \pm \sqrt{2}}{2}$	$\cos\theta = \dfrac{2 \pm \sqrt{2}}{2}$

There are no differences between the two solutions up to this point. However, for the algebraic equation, the solution is complete. There are two values of x that make the equation true: $x = \frac{2 + \sqrt{2}}{2}$ or $x = \frac{2 - \sqrt{2}}{2}$.

For the trigonometric equation, there appear to be two values of $\cos\theta$. Can we find values of θ for each of these two values of $\cos\theta$?

CASE 1 $\cos\theta = \dfrac{2 + \sqrt{2}}{2} \approx \dfrac{1 + 1.414}{2} = 1.207$

There is no value of θ such that $\cos\theta > 1$.

CASE 2 $\cos\theta = \dfrac{2 - \sqrt{2}}{2} \approx \dfrac{2 - 1.414}{2} = 0.293$

There are values of θ in the first quadrant and in the fourth quadrant such that $\cos\theta$ is a positive number less than 1. Use a calculator to approximate these values to the nearest degree.

ENTER: `2nd` `COS⁻¹` `(` 2 `−` `2nd` `√` 2 `)` `)` `÷` 2 `)` `ENTER`

DISPLAY:
```
cos⁻¹((2-√(2))/2)
         72.96875154
```

To the nearest degree, the value of θ in the first quadrant is 73°. This is also the value of the reference angle. Therefore, in the fourth quadrant, $\theta = 360° - 73°$ or 287°.

In the interval $0° \leq \theta < 360°$, the solution set of $2\cos^2\theta - 4\cos\theta + 1 = 0$ is:

$$\{73°, 287°\}$$

EXAMPLE 1

a. Use three different methods to solve $\tan^2\theta - 1 = 0$ for $\tan\theta$.

b. Find all possible values of θ in the interval $0 \le \theta < 2\pi$.

Solution **a.**

METHOD 1: FACTOR

$$\tan^2\theta - 1 = 0$$
$$(\tan\theta + 1)(\tan\theta - 1) = 0$$

$\tan\theta + 1 = 0$	$\tan\theta - 1 = 0$
$\tan\theta = -1$	$\tan\theta = 1$

METHOD 2: SQUARE ROOT

$$\tan^2\theta - 1 = 0$$
$$\tan^2\theta = 1$$
$$\tan\theta = \pm 1$$

METHOD 3: QUADRATIC FORMULA

$$\tan^2\theta - 1 = 0$$
$$a = 1, b = 0, c = -1$$
$$\tan\theta = \frac{-0 \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)} = \frac{\pm\sqrt{4}}{2} = \frac{\pm 2}{2} = \pm 1$$

b. When $\tan\theta = 1$, θ is in quadrant I or in quadrant III.

In quadrant I: if $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

This is also the measure of the reference angle.

In quadrant III: $\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

When $\tan\theta = -1$, θ is in quadrant II or in quadrant IV.

One value of θ is $-\frac{\pi}{4}$. The reference angle is $\frac{\pi}{4}$.

In quadrant II: $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

In quadrant IV: $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

Answer $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

EXAMPLE 2

Find, to the nearest degree, all possible values of B such that:

$$3\sin^2 B + 3\sin B - 2 = 0$$

Solution

$$a = 3, b = 3, c = -2$$
$$\sin B = \frac{-3 \pm \sqrt{(3)^2 - 4(3)(-2)}}{2(3)} = \frac{-3 \pm \sqrt{9 + 24}}{6} = \frac{-3 \pm \sqrt{33}}{6}$$

CASE 1 Let $\sin B = \frac{-3 + \sqrt{33}}{6}$.

Since $\frac{-3 + \sqrt{33}}{6} \approx 0.46$ is a number between -1 and 1, it is in the range of the sine function.

ENTER: 2nd SIN^{-1} (-3 + 2nd √ 33)) ÷ 6) ENTER

DISPLAY:
```
sin⁻¹((-3+√(33))
/6)        27.22120768
```

The sine function is positive in the first and second quadrants.

In quadrant I: $B = 27°$

In quadrant II: $B = 180 - 27 = 153°$

CASE 2 Let $\sin B = \frac{-3 - \sqrt{33}}{6}$.

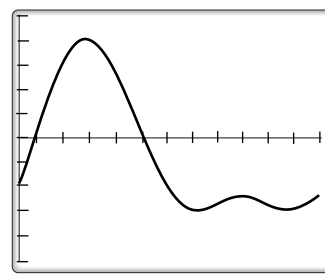
Since $\frac{-3 - \sqrt{33}}{6} \approx -1.46$ is not a number between -1 and 1, it is not in the range of the sine function. There are no values of B such that $\sin B = \frac{-3 - \sqrt{33}}{6}$. ✘

Calculator Solution Enter $Y_1 = 3\sin^2 X + 3\sin X - 2$ into the Y= menu.

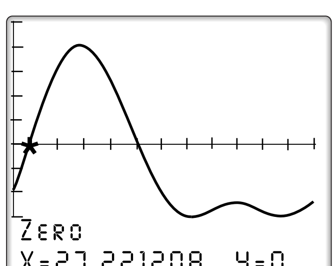
ENTER: Y= 3 SIN X,T,Θ,*n*) x^2 + 3 SIN X,T,Θ,*n*) − 2

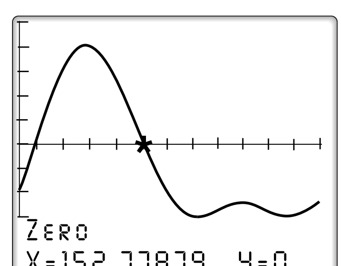
With the calculator set to DEGREE mode, graph the function in the following viewing window:

Xmin = 0, *Xmax* = 360, *Xscl* = 30, *Ymin* = −5, *Ymax* = 5



The solutions are the x-coordinates of the x-intercepts, that is, the roots of Y_1. Use the zero function of your graphing calculator to find the roots. Press 2nd CALC 2. Use the arrows to enter a left bound to the left of one of the zero values, a right bound to the right of the zero value, and a guess near the zero value. The calculator will display the coordinates of the point at which the graph intersects the x-axis. Repeat to find the other root.





As before, we find that the solutions in the interval $0° \leq \theta < 360°$ are approximately 27° and 153°.

Answer $B = 27° + 360n$ or $B = 153° + 360n$ for integral values of n.

Exercises

Writing About Mathematics

1. The discriminant of the quadratic equation $\tan^2 \theta + 4 \tan \theta + 5 = 0$ is -4. Explain why the solution set of this equation is the empty set.

2. Explain why the solution set of $2 \csc^2 \theta - \csc \theta = 0$ is the empty set.

Developing Skills

In 3–14, use the quadratic formula to find, to the nearest degree, all values of θ in the interval $0° \leq \theta < 360°$ that satisfy each equation.

3. $3 \sin^2 \theta - 7 \sin \theta - 3 = 0$

4. $\tan^2 \theta - 2 \tan \theta - 5 = 0$

5. $7 \cos^2 \theta - 1 = 5 \cos \theta$

6. $9 \sin^2 \theta + 6 \sin \theta = 2$

7. $\tan^2 \theta + 3 \tan \theta + 1 = 0$

8. $8 \cos^2 \theta - 7 \cos \theta + 1 = 0$

9. $2 \cot^2 \theta + 3 \cot \theta - 4 = 0$

10. $\sec^2 \theta - 2 \sec \theta - 4 = 0$

11. $3 \csc^2 \theta - 2 \csc \theta = 2$

12. $2 \tan \theta (\tan \theta + 1) = 3$

13. $3 \cos \theta + 1 = \frac{1}{\cos \theta}$

14. $\frac{\sin \theta}{2} = \frac{3}{\sin \theta + 2}$

15. Find all radian values of θ in the interval $0 \leq \theta < 2\pi$ for which $\frac{\sin \theta}{1} = \frac{1}{2 \sin \theta}$.

16. Find, to the nearest hundredth of a radian, all values of θ in the interval $0 \leq \theta < 2\pi$ for which $\frac{\cos \theta}{3} = \frac{1}{3 \cos \theta + 1}$.

13-4 USING SUBSTITUTION TO SOLVE TRIGONOMETRIC EQUATIONS INVOLVING MORE THAN ONE FUNCTION

When an equation contains two different functions, it may be possible to factor in order to write two equations, each with a different function. We can also use identities to write an equivalent equation with one function.

The equation $\cos^2 \theta + \sin \theta = 1$ cannot be solved by factoring. We can use the identity $\cos^2 \theta + \sin^2 \theta = 1$ to change the equation to an equivalent equation in $\sin \theta$ by replacing $\cos^2 \theta$ with $1 - \sin^2 \theta$.

$$\begin{aligned}
\cos^2 \theta + \sin \theta &= 1 \\
1 - \sin^2 \theta + \sin \theta &= 1 \\
-\sin^2 \theta + \sin \theta &= 0 \\
\sin \theta (-\sin \theta + 1) &= 0
\end{aligned}$$

$$\begin{array}{r|r}
\sin \theta = 0 & -\sin \theta + 1 = 0 \\
\theta_1 = 0° & 1 = \sin \theta \\
\theta_2 = 180° & \theta_3 = 90°
\end{array}$$

Check $\theta_1 = 0°$	*Check* $\theta_2 = 180°$	*Check* $\theta_3 = 90°$
$\cos^2\theta + \sin\theta = 1$	$\cos^2\theta + \sin\theta = 1$	$\cos^2\theta + \sin\theta = 1$
$\cos^2 0° + \sin 0° \stackrel{?}{=} 1$	$\cos^2 180° + \sin 180° \stackrel{?}{=} 1$	$\cos^2 90° + \sin 90° \stackrel{?}{=} 1$
$1^2 + 0 \stackrel{?}{=} 1$	$(-1)^2 + 0 \stackrel{?}{=} 1$	$0^2 + 1 \stackrel{?}{=} 1$
$1 = 1$ ✔	$1 = 1$ ✔	$1 = 1$ ✔

In the interval $0° \leq \theta < 360°$, the solution set of $\cos^2\theta + \sin\theta = 1$ is $\{0°, 90°, 180°\}$.

Is it possible to solve the equation $\cos^2\theta + \sin\theta = 1$ by writing an equivalent equation in terms of $\cos\theta$? To do so we must use an identity to write $\sin\theta$ in terms of $\cos\theta$. Since $\sin^2\theta = 1 - \cos^2\theta$, $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$.

(1) Write the equation: $\cos^2\theta + \sin\theta = 1$

(2) Replace $\sin\theta$ with $\pm\sqrt{1 - \cos^2\theta}$: $\cos^2\theta \pm\sqrt{1 - \cos^2\theta} = 1$

(3) Isolate the radical: $\pm\sqrt{1 - \cos^2\theta} = 1 - \cos^2\theta$

(4) Square both sides of the equation: $1 - \cos^2\theta = 1 - 2\cos^2\theta + \cos^4\theta$

(5) Write an equivalent equation with the right side equal to 0: $\cos^2\theta - \cos^4\theta = 0$

(6) Factor the left side: $\cos^2\theta\,(1 - \cos^2\theta) = 0$

(7) Set each factor equal to 0:

$\cos^2\theta = 0$	$1 - \cos^2\theta = 0$
$\cos\theta = 0$	$1 = \cos^2\theta$
$\theta_1 = 90°$	$\pm 1 = \cos\theta$
$\theta_2 = 270°$	$\theta_3 = 0°$
	$\theta_4 = 180°$

This approach uses more steps than the first. In addition, because it involves squaring both sides of the equation, an extraneous root, 270°, has been introduced. Note that 270° is a root of the equation $\cos^2\theta - \cos^4\theta = 0$ but is *not* a root of the given equation.

$$\cos^2\theta + \sin\theta = 1$$
$$\cos^2 270° + \sin 270° \stackrel{?}{=} 1$$
$$(0)^2 + (-1) \stackrel{?}{=} 1$$
$$0 - 1 \neq 1 \text{ ✘}$$

Any of the eight basic identities or the related identities can be substituted in a given equation.

EXAMPLE 1

Find all values of A in the interval $0° \leq A < 360°$ such that $2 \sin A + 1 = \csc A$.

Solution *How to Proceed*

(1) Write the equation: $$2 \sin A + 1 = \csc A$$

(2) Replace $\csc A$ with $\frac{1}{\sin A}$: $$2 \sin A + 1 = \frac{1}{\sin A}$$

(3) Multiply both sides of the equation by $\sin A$: $$2 \sin^2 A + \sin A = 1$$

(4) Write an equivalent equation with 0 as the right side: $$2 \sin^2 A + \sin A - 1 = 0$$

(5) Factor the left side: $$(2 \sin A - 1)(\sin A + 1) = 0$$

(6) Set each factor equal to 0 and solve for $\sin A$:

$$\begin{array}{r|r} 2 \sin A - 1 = 0 & \sin A + 1 = 0 \\ 2 \sin A = 1 & \sin A = -1 \\ \sin A = \frac{1}{2} & A = 270° \\ A = 30° & \\ \text{or } A = 150° & \end{array}$$

Answer $A = 30°$ or $A = 150°$ or $A = 270°$

Often, more than one substitution is necessary to solve an equation.

EXAMPLE 2

If $0 \leq \theta < 2\pi$, find the solution set of the equation $2 \sin \theta = 3 \cot \theta$.

Solution *How to Proceed*

(1) Write the equation: $$2 \sin \theta = 3 \cot \theta$$

(2) Replace $\cot \theta$ with $\frac{\cos \theta}{\sin \theta}$: $$2 \sin \theta = 3\left(\frac{\cos \theta}{\sin \theta}\right)$$

(3) Multiply both sides of the equation by $\sin \theta$: $$2 \sin^2 \theta = 3 \cos \theta$$

(4) Replace $\sin^2 \theta$ with $1 - \cos^2 \theta$: $$2(1 - \cos^2 \theta) = 3 \cos \theta$$

(5) Write an equivalent equation in standard form:

$$2 - 2 \cos^2 \theta = 3 \cos \theta$$
$$2 - 2 \cos^2 \theta - 3 \cos \theta = 0$$
$$2 \cos^2 \theta + 3 \cos \theta - 2 = 0$$

(6) Factor and solve for $\cos\theta$: $(2\cos\theta - 1)(\cos\theta + 2) = 0$

(7) Find all values of θ in the given interval:

$2\cos\theta - 1 = 0$	$\cos\theta + 2 = 0$
$2\cos\theta = 1$	$\cos\theta = -2$ ✘
$\cos\theta = \frac{1}{2}$	No solution
$\theta = \frac{\pi}{3}$	
or $\theta = \frac{5\pi}{3}$	

Answer $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

The following identities from Chapter 10 will be useful in solving trigonometric equations:

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cos^2\theta + \sin^2\theta = 1$
$\csc\theta = \frac{1}{\sin\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$1 + \tan^2\theta = \sec^2\theta$
$\cot\theta = \frac{1}{\tan\theta}$		$\cot^2\theta + 1 = \csc^2\theta$

Exercises

Writing About Mathematics

1. Sasha said that $\sin\theta + \cos\theta = 2$ has no solution. Do you agree with Sasha? Explain why or why not.

2. For what values of θ is $\sin\theta = \sqrt{1 - \cos^2\theta}$ true?

Developing Skills

In 3–14, find the exact values of θ in the interval $0° \leq \theta < 360°$ that satisfy each equation.

3. $2\cos^2\theta - 3\sin\theta = 0$

4. $4\cos^2\theta + 4\sin\theta - 5 = 0$

5. $\csc^2\theta - \cot\theta - 1 = 0$

6. $2\sin\theta + 1 = \csc\theta$

7. $2\sin^2\theta + 3\cos\theta - 3 = 0$

8. $3\tan\theta = \cot\theta$

9. $2\cos\theta = \sec\theta$

10. $\sin\theta = \csc\theta$

11. $\tan\theta = \cot\theta$

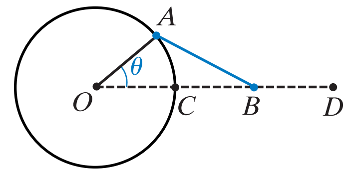
12. $2\cos^2\theta = \sin\theta + 2$

13. $\cot^2\theta = \csc\theta + 1$

14. $2\sin^2\theta - \tan\theta\cot\theta = 0$

Applying Skills

15. An engineer would like to model a piece for a factory machine on his computer. As shown in the figure, the machine consists of a link fixed to a circle at point A. The other end of the link is fixed to a slider at point B. As the circle rotates, point B slides back and forth between the two ends of the slider (C and D). The movement is restricted so that θ, the measure of $\angle AOD$, is in the interval $-45° \leq \theta \leq 45°$. The motion of point B can be described mathematically by the formula



$$CB = r(\cos\theta - 1) + \sqrt{l^2 - r^2\sin^2\theta}$$

where r is the radius of the circle and l is the length of the link. Both the radius of the circle and the length of the link are 2 inches.

a. Find the exact value of CB when: (1) $\theta = 30°$ (2) $\theta = 45°$.

b. Find the exact value(s) of θ when $CB = 2$ inches.

c. Find, to the nearest hundredth of a degree, the value(s) of θ when $CB = 1.5$ inches.

13-5 USING SUBSTITUTION TO SOLVE TRIGONOMETRIC EQUATIONS INVOLVING DIFFERENT ANGLE MEASURES

If an equation contains function values of two different but related angle measures, we can use identities to write an equivalent equation in terms of just one variable. For example: Find the value(s) of θ such that $\sin 2\theta - \sin\theta = 0$.

Recall that $\sin 2\theta = 2\sin\theta\cos\theta$. We can use this identity to write the equation in terms of just one variable, θ, and then use any convenient method to solve the equation.

(1) Write the equation: $\quad \sin 2\theta - \sin\theta = 0$

(2) For $\sin 2\theta$, substitute its equal, $2\sin\theta\cos\theta$: $\quad 2\sin\theta\cos\theta - \sin\theta = 0$

(3) Factor the left side: $\quad \sin\theta(2\cos\theta - 1) = 0$

(4) Set each factor equal to zero and solve for θ:

$\sin\theta = 0$	$2\cos\theta - 1 = \theta$
$\theta = 0°$	$2\cos\theta = 1$
or $\theta = 180°$	$\cos\theta = \frac{1}{2}$
	$\theta = 60°$
	or $\theta = 300°$

The degree measures $0°, 60°, 180°$, and $300°$ are all of the values in the interval $0° \leq \theta < 360°$ that make the equation true. Any values that differ from these values by a multiple of $360°$ will also make the equation true.

EXAMPLE 1

Find, to the nearest degree, the roots of $\cos 2\theta - 2\cos\theta = 0$.

Solution *How to Proceed*

(1) Write the given equation:

$$\cos 2\theta - 2\cos\theta = 0$$

(2) Use an identity to write $\cos 2\theta$ in terms of $\cos\theta$:

$$2\cos^2\theta - 1 - 2\cos\theta = 0$$

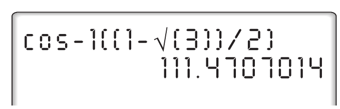
(3) Write the equation in standard form:

$$2\cos^2\theta - 2\cos\theta - 1 = 0$$

(4) The equation cannot be factored over the set of integers. Use the quadratic formula:

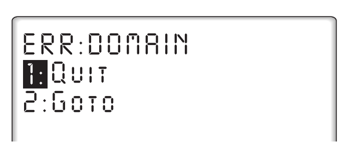
$$\cos\theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$
$$= \frac{2 \pm \sqrt{12}}{4}$$
$$= \frac{1 \pm \sqrt{3}}{2}$$

(5) When we use a calculator to approximate the value of $\arccos\frac{1 - \sqrt{3}}{2}$, the calculator will return the value 111°. Cosine is negative in both the second and the third quadrants. Therefore, there is both a second-quadrant and a third-quadrant angle such that $\cos\theta = \frac{1 - \sqrt{3}}{2}$:



$$\theta_1 = 111°$$
$$R = 180° - 111° = 69°$$
$$\theta_2 = 180° + 69° = 249°$$

(6) When we use a calculator to approximate the value of $\arccos\frac{1 + \sqrt{3}}{2}$, the calculator will return an error message because $\frac{1 + \sqrt{3}}{2} > 1$ is not in the domain of arccosine:



Answer To the nearest degree, $\theta = 111°$ or $\theta = 249°$.

EXAMPLE 2

Find, to the nearest degree, the values of θ in the interval $0° \leq \theta \leq 360°$ that are solutions of the equation $\sin(90° - \theta) + 2\cos\theta = 2$.

Solution Use the identity $\sin(90° - \theta) = \cos\theta$.

$$\begin{aligned} \sin(90° - \theta) + 2\cos\theta &= 2 \\ \cos\theta + 2\cos\theta &= 2 \\ 3\cos\theta &= 2 \\ \cos\theta &= \tfrac{2}{3} \end{aligned}$$

To the nearest degree, a calculator returns the value of θ as 48°.

In quadrant I, $\theta = 48°$ and in quadrant IV, $\theta = 360° - 48° = 312°$.

Answer $\theta = 48°$ or $\theta = 312°$

The basic trigonometric identities along with the cofunction, double-angle, and half-angle identities will be useful in solving trigonometric equations:

Cofunction Identities	Double-Angle Identities	Half-Angle Identities
$\cos\theta = \sin(90° - \theta)$	$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}$
$\sin\theta = \cos(90° - \theta)$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$	$\cos\frac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}$
$\tan\theta = \cot(90° - \theta)$	$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$	$\tan\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$
$\cot\theta = \tan(90° - \theta)$		
$\sec\theta = \csc(90° - \theta)$		
$\csc\theta = \sec(90° - \theta)$		

Note that in radians, the right sides of the cofunction identities are written in terms of $\pi - \theta$.

Exercises

Writing About Mathematics

1. Isaiah said that if the equation $\cos 2x + 2\cos^2 x = 2$ is divided by 2, an equivalent equation is $\cos x + \cos^2 x = 1$. Do you agree with Isaiah? Explain why or why not.

2. Aaron solved the equation $2\sin\theta\cos\theta = \cos\theta$ by first dividing both sides of the equation by $\cos\theta$. Aaron said that for $0 \leq \theta \leq 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$. Do you agree with Aaron? Explain why or why not.

Developing Skills

In 3–10, find the exact values of θ in the interval $0° \leq \theta \leq 360°$ that make each equation true.

3. $\sin 2\theta - \cos \theta = 0$

4. $\cos 2\theta + \sin^2 \theta = 1$

5. $\sin 2\theta + 2 \sin \theta = 0$

6. $\tan 2\theta = \cot \theta$

7. $\cos 2\theta + 2 \cos^2 \theta = 2$

8. $\sin \frac{1}{2}\theta = \cos \theta$

9. $3 - 3 \sin \theta - 2 \cos^2 \theta = 0$

10. $3 \cos 2\theta - 4 \cos^2 \theta + 2 = 0$

In 11–18, find all radian measures of θ in the interval $0 \leq \theta \leq 2\pi$ that make each equation true. Express your answers in terms of π when possible; otherwise, to the nearest hundredth.

11. $\cos 2\theta = 2 \cos \theta - 2 \cos^2 \theta$

12. $2 \sin 2\theta + \sin \theta = 0$

13. $5 \sin^2 \theta - 4 \sin \theta + \cos 2\theta = 0$

14. $\cos \theta = 3 \sin 2\theta$

15. $3 \sin 2\theta = \tan \theta$

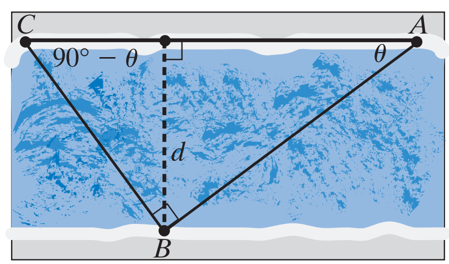
16. $\sin (90° - \theta) + \cos^2 \theta = \frac{1}{4}$

17. $2 \cos^2 \theta + 3 \sin \theta - 2 \cos 2\theta = 1$

18. $(2 \sin \theta \cos \theta)^2 + 4 \sin 2\theta - 1 = 0$

Applying Skills

19. Martha swims 90 meters from point A on the north bank of a stream to point B on the opposite bank. Then she makes a right angle turn and swims 60 meters from point B to point C, another point on the north bank. If $m\angle CAB = \theta$, then $m\angle ACB = 90° - \theta$.



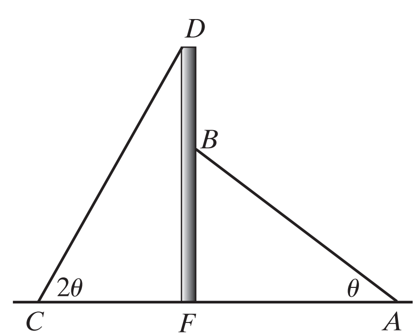
a. Let d be the width of the stream, the length of the perpendicular distance from B to $\overline{AC}$. Express d in terms of $\sin \theta$.

b. Express d in terms of $\sin (90° - \theta)$.

c. Use the answers to **a** and **b** to write an equation. Solve the equation for θ.

d. Find d, the width of the stream.

20. A pole is braced by two wires of equal length as shown in the diagram. One wire, $\overline{AB}$, makes an angle of θ with the ground, and the other wire, $\overline{CD}$, makes an angle of 2θ with the ground. If $FD = 1.75FB$, find, to the nearest degree, the measure of θ:



a. Let $AB = CD = x$, $FB = y$, and $FD = 1.75y$. Express $\sin \theta$ and $\sin 2\theta$ in terms of x and y.

b. Write an equation that expresses a relationship between $\sin \theta$ and $\sin 2\theta$ and solve for θ to the nearest degree.

CHAPTER SUMMARY

A **trigonometric equation** is an equation whose variable is expressed in terms of a trigonometric function value.

The following table compares the degree measures of θ from $-90°$ to $360°$, the radian measures of θ from $-\frac{\pi}{2}$ to 2π, and the measure of its reference angle.

	Fourth Quadrant	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
Angle	$-90° < \theta < 0°$ $-\frac{\pi}{2} < \theta < 0$	$0° < \theta < 90°$ $0 < \theta < \frac{\pi}{2}$	$90° < \theta < 180°$ $\frac{\pi}{2} < \theta < \theta$	$180° < \theta < 270°$ $\pi < \theta < \frac{3\pi}{2}$	$270° < \theta < 360°$ $\frac{3\pi}{2} < \theta < 2\pi$
Reference Angle	$-\theta$ $-\theta$	θ θ	$180° - \theta$ $\pi - \theta$	$\theta - 180°$ $\theta - \pi$	$360° - \theta$ $2\pi - \theta$

To solve a trigonometric equation:

1. If the equation involves more than one variable, use identities to write the equation in terms of one variable.
2. If the equation involves more than one trigonometric function of the same variable, separate the functions by factoring or use identities to write the equation in terms of one function of one variable.
3. Solve the equation for the function value of the variable. Use factoring or the quadratic formula to solve a second-degree equation.
4. Use a calculator or your knowledge of the exact function values to write one value of the variable to an acceptable degree of accuracy.
5. If the measure of the angle found in step 4 is not that of a quadrantal angle, find the measure of its reference angle.
6. Use the measure of the reference angle to find the degree measures of each solution in the interval $0° \leq \theta < 360°$ or the radian measures of each solution in the interval $0 \leq \theta < 2\pi$.
7. Add $360n$ (n an integer) to the solutions in degrees found in steps 4 and 6 to write all possible solutions in degrees. Add $2\pi n$ (n an integer) to the solutions in radians found in steps 2 and 4 to write all possible solutions in radians.

VOCABULARY

13-1 Trigonometric equation

REVIEW EXERCISES

In 1–10, find the exact values of x in the interval $0° \leq x \leq 360°$ that make each equation true.

1. $2 \cos x + 1 = 0$

2. $\sqrt{3} - \sin x = \sin x + \sqrt{12}$

3. $2 \sec x = 2 + \sec x$

4. $2 \cos^2 x + \cos x - 1 = 0$

5. $\cos x \sin x + \sin x = 0$

6. $\tan x - 3 \cot x = 0$

7. $2 \cos x - \sec x = 0$

8. $\sin^2 x - \cos^2 x = 0$

9. $2 \tan x = 1 - \tan^2 x$

10. $\cos^3 x - \frac{3}{4} \cos x = 0$

In 11–22, find, to the nearest hundredth, all values of θ in the interval $0 \leq \theta < 2\pi$ that make each equation true.

11. $7 \sin \theta + 3 = 1$

12. $5(\cos \theta - 1) = 6 + \cos \theta$

13. $4 \sin^2 \theta - 3 \sin \theta = 1$

14. $3 \cos^2 \theta - \cos \theta - 2 = 0$

15. $\tan^2 \theta - 4 \tan \theta - 1 = 0$

16. $\sec^2 \theta - 10 \sec \theta + 20 = 0$

17. $\tan 2\theta = 4 \tan \theta$

18. $2 \sin 2\theta + \cos \theta = 0$

19. $\frac{\sin 2\theta}{1 + \cos 2\theta} = 4$

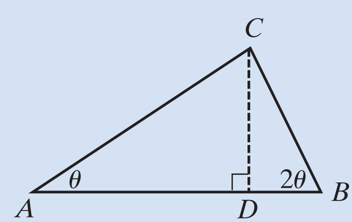
20. $3 \cos 2\theta + \cos \theta + 2 = 0$

21. $2 \tan^2 \theta + 6 \tan \theta = 20$

22. $\cos 2\theta - \cos^2 \theta + \cos \theta + \frac{1}{4} = 0$

23. Explain why the solution set of $\tan \theta - \sec \theta = 0$ is the empty set.

24. In $\triangle ABC$, $m\angle A = \theta$ and $m\angle B = 2\theta$. The altitude from C intersects $\overline{AB}$ at D and $AD : DB = 5 : 2$.



a. Write $\tan \theta$ and $\tan 2\theta$ as ratios of the sides of $\triangle ADC$ and $\triangle BDC$, respectively.

b. Solve the equation for $\tan 2\theta$ found in **a** for CD.

c. Substitute the value of CD found in **b** into the equation of $\tan \theta$ found in **a**.

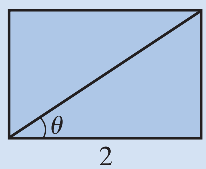
d. Solve for θ.

e. Find the measures of the angles of the triangle to the nearest degree.

Exploration

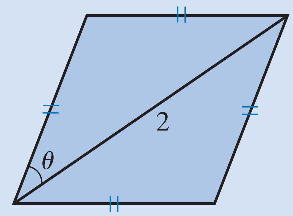
For (1)–(6): **a.** Use your knowledge of geometry and trigonometry to express the area, A, of the shaded region in terms of θ. **b.** Find the measure of θ when $A = 0.5$ square unit or explain why there is no possible value of θ. Give the exact value when possible; otherwise, to the nearest hundredth of a radian.

(1)



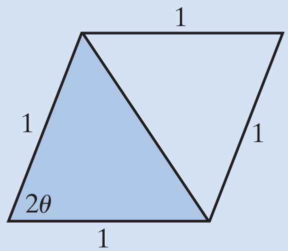
$0 < \theta < \frac{\pi}{2}$

(2)



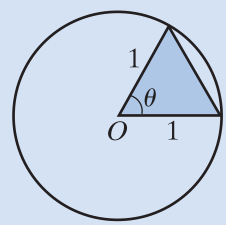
$0 < \theta < \frac{\pi}{2}$

(3)



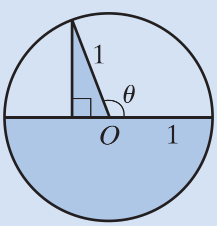
$0 < \theta < \frac{\pi}{2}$

(4)



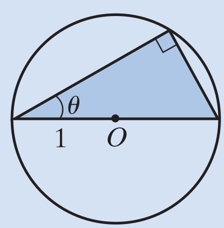
$0 < \theta < \pi$

(5)



$\frac{\pi}{2} < \theta \leq \pi$

(6)



$0 < \theta < \frac{\pi}{2}$

CUMULATIVE REVIEW CHAPTERS 1–13

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The expression $2(5)^0 + 3(27)^{-\frac{1}{3}}$ is equal to

(1) -27 (2) $2 + \frac{\sqrt{3}}{9}$ (3) 3 (4) $2\frac{1}{9}$

2. The sum of $\frac{3}{4}a^2 - \frac{1}{2}a$ and $a - \frac{1}{2}a^2$ is

(1) $\frac{1}{4}a^2 + \frac{1}{2}a$ (3) $\frac{5}{4}a^2 - \frac{3}{2}a$

(2) $2a^3$ (4) $\frac{7}{4}a^2 - a$

3. The fraction $\frac{4}{1 - \sqrt{5}}$ is equivalent to

(1) $1 + \sqrt{5}$ (3) $-1 + \sqrt{5}$

(2) $1 - \sqrt{5}$ (4) $-1 - \sqrt{5}$

4. The complex number $i^{12} + i^{10}$ can be written as

(1) 0 (2) 1 (3) 2 (4) $1 + i$

5. $\sum_{n=1}^{4}\left[(-1)^n \frac{n}{2}\right]$ is equal to

(1) -5 (2) 1 (3) 2 (4) 5

6. When the roots of a quadratic equation are real and irrational, the discriminant must be

(1) zero.

(2) a positive number that is a perfect square.

(3) a positive number that is not a perfect square.

(4) a negative number.

7. When $f(x) = x^2 + 1$ and $g(x) = 2x$, then $g(f(x))$ equals

(1) $4x^2 + 1$ (3) $4x^2 + 2$

(2) $2x^2 + 2x + 1$ (4) $2x^2 + 2$

8. If $\log x = 2 \log a - \frac{1}{3} \log b$, then x equals

(1) $2a - \frac{1}{3}b$ (3) $\frac{a^2}{\frac{1}{3}b}$

(2) $a^2 - \sqrt[3]{b}$ (4) $\frac{a^2}{\sqrt[3]{b}}$

9. What number must be added to the binomial $x^2 + 5x$ in order to change it into a trinomial that is a perfect square?

(1) $\frac{5}{2}$ (2) $\frac{25}{4}$ (3) $\frac{25}{2}$ (4) 25

10. If f(x) is a function from the set of real numbers to the set of real numbers, which of the following functions is one-to-one and onto?

(1) $f(x) = 2x - 1$ (3) $f(x) = |2x - 1|$

(2) $f(x) = x^2$ (4) $f(x) = -x^2 + 2x$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Sketch the graph of the inequality $y \leq -x^2 - 2x + 3$.

12. When $f(x) = 4x - 2$, find $f^{-1}(x)$, the inverse of $f(x)$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Write the first six terms of the geometric function whose first term is 2 and whose fourth term is 18.

14. The endpoints of a diameter of a circle are $(-2, 5)$ and $(4, -1)$. Write an equation of the circle.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. If $\csc \theta = 3$, and $\cos \theta < 0$, find $\sin \theta$, $\cos \theta$, $\tan \theta$, $\cot \theta$ and $\sec \theta$.

16. Solve for x and write the roots in $a + bi$ form: $\frac{6}{x} - 2 = \frac{5}{x^2}$.