CHAPTER

12

TRIGONOMETRIC IDENTITIES

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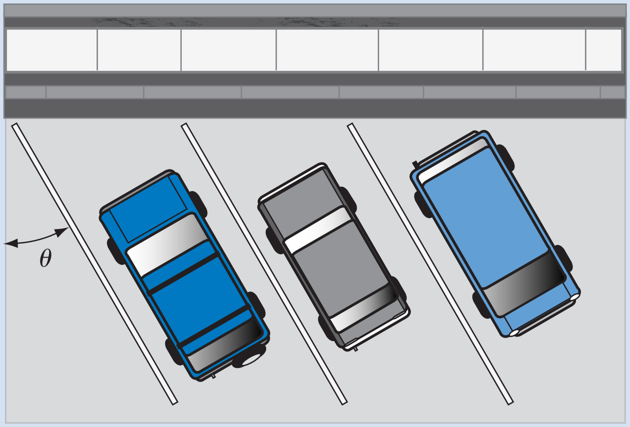
When a busy street passes through the business center of a town, merchants want to insure maximum parking in order to make stopping to shop convenient. The town planners must decide whether to allow parallel parking (parking parallel to the curb) on both sides of the street or angle parking (parking at an angle with the curb). The size of angle the parking space makes with the curb determines the amount of road space needed for parking and may limit parking to only one side of the street. Problems such as this require the use of function values of the parking angles and illustrate one way trigonometric identities help us to solve problems.



12-1 BASIC IDENTITIES

Throughout our study of mathematics, we have used the solution of equations to solve problems. The domain of an equation is the set of numbers for which each side of the equation is defined. When the solution set is a proper subset of the domain, the equation is a *conditional equation*. When the solution set is the domain of the equation, the equation is an *identity*.

Recall that an identity is an equation that is true for all possible replacements of the variable.

In the following examples, the domain is the set of real numbers.

Algebraic Equations

Conditional equation:
$x^2 + 3x - 10 = 0$
Solution set: $\{-5, 2\}$

Identity:
$3x + 12 = 3(x + 4)$
Solution set: {Real numbers}

Trigonometric Equations

Conditional equation:
$\sin\theta = 1$
Solution set: $\left\{\frac{\pi}{2} + 2\pi n, n = \text{an integer}\right\}$

Identity:
$\sin^2\theta + \cos^2\theta = 1$
Solution set: {Real numbers}

When stated in the form of an equation, a property of the real numbers or an application of a property of the real numbers is an algebraic identity. For example, the additive identity property can be expressed as an algebraic identity: $a + 0 = a$ is true for all real numbers.

When we defined the six trigonometric functions, we proved relationships that are true for all values of θ for which the function is defined. There are eight basic trigonometric identities.

Pythagorean Identities	Reciprocal Identities	Quotient Identities
$\cos^2\theta + \sin^2\theta = 1$	$\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$1 + \tan^2\theta = \sec^2\theta$	$\csc\theta = \frac{1}{\sin\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\cot^2\theta + 1 = \csc^2\theta$	$\cot\theta = \frac{1}{\tan\theta}$	

Each of these identities is true for all values of θ for which both sides of the identity are defined. For example, $\cos^2\theta + \sin^2\theta = 1$ is true for all real numbers and $1 + \tan^2\theta = \sec^2\theta$ is true for all real numbers except $\theta = \frac{\pi}{2} + n\pi$ when n is an integer.

We can use the eight basic identities to write other equations that are true for all replacements of the variable for which the function values exist.

EXAMPLE 1

Use the basic identities to show that $\tan\theta \csc\theta = \sec\theta$ for all values of θ for which each side of the equation is defined.

Solution Each of the functions in the given equation can be written in terms of $\sin\theta$, $\cos\theta$, or both.

$$\tan\theta \csc\theta \stackrel{?}{=} \sec\theta$$

(1) Use the basic identities to write each side of the identity in terms of $\sin\theta$ and $\cos\theta$:

$$\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right) \stackrel{?}{=} \frac{1}{\cos\theta}$$

(2) Divide a numerator and a denominator of the left side of the equation by $\sin\theta$:

$$\left(\frac{\overset{1}{\cancel{\sin\theta}}}{\cos\theta}\right)\left(\frac{1}{\underset{1}{\cancel{\sin\theta}}}\right) \stackrel{?}{=} \frac{1}{\cos\theta}$$

Note: If $\csc\theta$ is defined, $\sin\theta \neq 0$.

$$\frac{1}{\cos\theta} = \frac{1}{\cos\theta} \checkmark$$

EXAMPLE 2

Use the Pythagorean identities to write:

a. $\sin\theta$ in terms of $\cos\theta$. **b.** $\cos\theta$ in terms of $\sin\theta$.

Solution

a.
$$\cos^2\theta + \sin^2\theta = 1$$
$$\sin^2\theta = 1 - \cos^2\theta$$
$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

b.
$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

Note: For a given value of θ, the sign of $\cos\theta$ or of $\sin\theta$ depends on the quadrant in which the terminal side of the angle lies:

- When θ is a first-quadrant angle, $\sin\theta = \sqrt{1-\cos^2\theta}$ and $\cos\theta = \sqrt{1-\sin^2\theta}$.
- When θ is a second-quadrant angle, $\sin\theta = \sqrt{1-\cos^2\theta}$ and $\cos\theta = -\sqrt{1-\sin^2\theta}$.
- When θ is a third-quadrant angle, $\sin\theta = -\sqrt{1-\cos^2\theta}$ and $\cos\theta = -\sqrt{1-\sin^2\theta}$.
- When θ is a fourth-quadrant angle, $\sin\theta = -\sqrt{1-\cos^2\theta}$ and $\cos\theta = \sqrt{1-\sin^2\theta}$.

Exercises

Writing About Mathematics

1. If we know the value of $\sin\theta$, is it possible to find the other five trigonometric function values? If not, what other information is needed?

2. a. Explain how the identities $1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$ can be derived from the identity $\cos^2\theta + \sin^2\theta = 1$.

b. The identity $\cos^2\theta + \sin^2\theta = 1$ is true for all real numbers. Are the identities $1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$ also true for all real numbers? Explain your answer.

Developing Skills

In 3–14, write each expression as a single term using $\sin\theta$, $\cos\theta$, or both.

3. $\tan\theta$

4. $\cot\theta$

5. $\sec\theta$

6. $\csc\theta$

7. $\cot\theta\sec\theta$

8. $\tan^2\theta + 1$

9. $\cot^2\theta + 1$

10. $\tan\theta\sec\theta\cot\theta$

11. $\frac{1}{\sec\theta\csc\theta}$

12. $\frac{\tan\theta}{\cot\theta} + \tan\theta\cot\theta$

13. $\frac{1}{\tan\theta} + \cot\theta$

14. $\sec\theta + \frac{1}{\csc\theta}$

12-2 PROVING AN IDENTITY

The eight basic identities are used to prove other identities. To prove an identity means to show that the two sides of the equation are always equivalent. It is generally more efficient to work with the more complicated side of the identity and show, by using the basic identities and algebraic principles, that the two sides are the same.

Tips for Proving an Identity

To prove an identity, use one or more of the following tips:

1. Work with the more complicated side of the equation.
2. Use basic identities to rewrite unlike functions in terms of the same function.
3. Remove parentheses.
4. Find common denominators to add fractions.
5. Simplify complex fractions and reduce fractions to lowest terms.

EXAMPLE 1

Prove that $\sec\theta\sin\theta = \tan\theta$ is an identity.

Solution Write the left side of the equation in terms of $\sin\theta$ and $\cos\theta$.

$$\begin{aligned} \sec\theta\sin\theta &\stackrel{?}{=} \tan\theta \\ \frac{1}{\cos\theta}\sin\theta &\stackrel{?}{=} \tan\theta \\ \frac{\sin\theta}{\cos\theta} &\stackrel{?}{=} \tan\theta \\ \tan\theta &= \tan\theta \checkmark \end{aligned}$$

Proof begins with what is known and proceeds to what is to be proved. Although we have written the proof in Example 1 by starting with what is to be proved and ending with what is obviously true, the proof of this identity really begins with the obviously true statement:

$$\tan\theta = \tan\theta; \text{ therefore, } \sec\theta\sin\theta = \tan\theta.$$

EXAMPLE 2

Prove that $\sin\theta\,(\csc\theta - \sin\theta) = \cos^2\theta$.

Solution Use the distributive property to simplify the left side.

$$\begin{aligned} \sin\theta\,(\csc\theta - \sin\theta) &\stackrel{?}{=} \cos^2\theta \\ \sin\theta\csc\theta - \sin^2\theta &\stackrel{?}{=} \cos^2\theta \\ \sin\theta\left(\frac{1}{\sin\theta}\right) - \sin^2\theta &\stackrel{?}{=} \cos^2\theta \\ 1 - \sin^2\theta &\stackrel{?}{=} \cos^2\theta \\ (\cos^2\theta + \sin^2\theta) - \sin^2\theta &\stackrel{?}{=} \cos^2\theta \\ \cos^2\theta &= \cos^2\theta \checkmark \end{aligned}$$

Use the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$.

The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ can be rewritten as $\cos^2\theta = 1 - \sin^2\theta$. The second to last line of the proof is often omitted and the left side, $1 - \sin^2\theta$, replaced by $\cos^2\theta$.

EXAMPLE 3

Prove the identity $1 - \sin\theta = \frac{\cos^2\theta}{1 + \sin\theta}$.

Solution For this identity, it appears that we need to multiply both sides of the equation by $(1 + \sin\theta)$ to clear the denominator. However, in proving an identity we perform only operations that change the form but not the value of that side of the equation.

Here we will work with the right side because it is more complicated and multiply by $\frac{1 - \sin\theta}{1 - \sin\theta}$, a fraction equal to 1.

$$1 - \sin\theta \stackrel{?}{=} \frac{\cos^2\theta}{1 + \sin\theta}$$
$$1 - \sin\theta \stackrel{?}{=} \frac{\cos^2\theta}{1 + \sin\theta} \times \frac{1 - \sin\theta}{1 - \sin\theta}$$
$$1 - \sin\theta \stackrel{?}{=} \frac{\cos^2\theta\,(1 - \sin\theta)}{1 - \sin^2\theta}$$
$$1 - \sin\theta \stackrel{?}{=} \frac{\cos^2\theta\,(1 - \sin\theta)}{\cos^2\theta}$$
$$1 - \sin\theta = 1 - \sin\theta \checkmark$$

EXAMPLE 4

Prove the identity $\frac{\cot^2\theta}{\csc\theta + 1} + 1 = \csc\theta$.

Solution In this identity we will work with the left side.

How to Proceed	
(1) Write the given equation:	$\frac{\cot^2\theta}{\csc\theta + 1} + 1 \stackrel{?}{=} \csc\theta$
(2) Write 1 as a fraction with the same denominator as the given fraction:	$\frac{\cot^2\theta}{\csc\theta + 1} + \frac{\csc\theta + 1}{\csc\theta + 1} \stackrel{?}{=} \csc\theta$
(3) Add the fractions:	$\frac{\cot^2\theta + \csc\theta + 1}{\csc\theta + 1} \stackrel{?}{=} \csc\theta$
(4) Use the identity $\cot^2\theta + 1 = \csc^2\theta$:	$\frac{\csc^2\theta + \csc\theta}{\csc\theta + 1} \stackrel{?}{=} \csc\theta$
(5) Factor the numerator:	$\frac{\csc\theta(\csc\theta + 1)}{\csc\theta + 1} \stackrel{?}{=} \csc\theta$
(6) Divide the numerator and denominator by $(\csc\theta + 1)$:	$\csc\theta = \csc\theta \checkmark$

Exercises

Writing About Mathematics

1. Is $\sin\theta = \sqrt{1 - \cos^2\theta}$ an identity? Explain why or why not.

2. Cory said that in Example 3, $1 - \sin\theta = \frac{\cos^2\theta}{1 + \sin\theta}$ could have been shown to be an identity by multiplying the left side by $\frac{1 + \sin\theta}{1 + \sin\theta}$. Do you agree with Cory? Explain why or why not.

Developing Skills

In 3–26, prove that each equation is an identity.

3. $\sin\theta \csc\theta \cos\theta = \cos\theta$

4. $\tan\theta \sin\theta \cos\theta = \sin^2\theta$

5. $\cot\theta \sin\theta \cos\theta = \cos^2\theta$

6. $\sec\theta(\cos\theta - \cot\theta) = 1 - \csc\theta$

7. $\csc\theta(\sin\theta + \tan\theta) = 1 + \sec\theta$

8. $1 - \frac{\cos\theta}{\sec\theta} = \sin^2\theta$

9. $1 - \frac{\sin\theta}{\csc\theta} = \cos^2\theta$

10. $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$

11. $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

12. $\frac{\tan\theta}{\sec\theta} = \sin\theta$

13. $\frac{\cot\theta}{\csc\theta} = \cos\theta$

14. $\frac{\csc\theta}{\sec\theta} = \cot\theta$

15. $\frac{\sec\theta}{\csc\theta} = \tan\theta$

16. $\frac{1}{\sin\theta\cos\theta} - \frac{\cos\theta}{\sin\theta} = \tan\theta$

17. $\frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$

18. $\frac{\sin^2\theta}{1 + \cos\theta} = 1 - \cos\theta$

19. $\frac{\cos^2\theta}{1 + \sin\theta} = 1 - \sin\theta$

20. $\sec\theta \csc\theta = \tan\theta + \cot\theta$

21. $\frac{\tan^2\theta}{\sec\theta - 1} - 1 = \sec\theta$

22. $\cos\theta + \frac{\sin^2\theta}{1 + \cos\theta} = 1$

23. $\sin\theta + \frac{\cos^2\theta}{1 + \sin\theta} = 1$

24. $\frac{\sec\theta}{\cos\theta} - \tan^2\theta = 1$

25. $\frac{\csc\theta}{\sin\theta} - \cot^2\theta = 1$

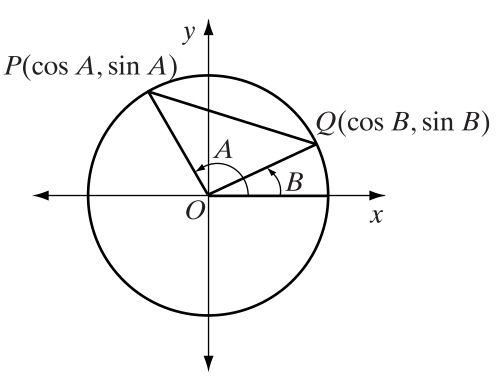
26. $\frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\csc\theta} = 1$

27. For what values of θ is the identity $\frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\csc\theta} = 1$ undefined?

12-3 COSINE $(A - B)$

We can prove that $\cos(A - B) = \cos A - \cos B$ is *not* an identity by finding one pair of values of A and B for which each side of the equation is defined and the equation is false. For example, if, in degree measure, $A = 90°$ and $B = 60°$,

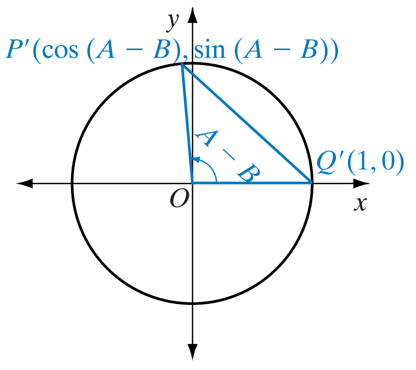
$$\cos(90° - 60°) \stackrel{?}{=} \cos 90° - \cos 60°$$
$$\cos 30° \stackrel{?}{=} 0 - \tfrac{1}{2}$$
$$\tfrac{\sqrt{3}}{2} \neq -\tfrac{1}{2} \text{ ✘}$$



In order to write an identity that expresses $\cos(A - B)$ in terms of function values of A and B, we will use the relationship between the unit circle and the sine and cosine of an angle.

Let A and B be any two angles in standard position. The terminal side of $\angle A$ intersects the unit circle at $P(\cos A, \sin A)$ and the terminal side of $\angle B$ intersects the unit circle at $Q(\cos B, \sin B)$. Use the distance formula to express PQ in terms of $\sin A$, $\cos A$, $\sin B$, and $\cos B$:

$$\begin{aligned} PQ^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= (\cos^2 A - 2\cos A \cos B + \cos^2 B) + (\sin^2 A - 2\sin A \sin B + \sin^2 B) \\ &= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2\cos A \cos B - 2\sin A \sin B \\ &= 1 + 1 - 2(\cos A \cos B + \sin A \sin B) \\ &= 2 - 2(\cos A \cos B + \sin A \sin B) \end{aligned}$$



Now rotate $\triangle OQP$ through an angle of $-B$, that is, an angle of B units in the clockwise direction so that the image of P is P' and the image of Q is Q'. Q' is a point on the x-axis whose coordinates are $(1, 0)$. Angle $Q'OP'$ is an angle in standard position whose measure is $(A - B)$. Therefore, the coordinates of P' are $(\cos(A - B), \sin(A - B))$. Use the distance formula to find $P'Q'$.

$$\begin{aligned} (P'Q')^2 &= (\cos(A - B) - 1)^2 + (\sin(A - B) - 0)^2 \\ &= \cos^2(A - B) - 2\cos(A - B) + 1 + \sin^2(A - B) \\ &= [\cos^2(A - B) + \sin^2(A - B)] - 2\cos(A - B) + 1 \\ &= 1 - 2\cos(A - B) + 1 \\ &= 2 - 2\cos(A - B) \end{aligned}$$

Distance is preserved under a rotation. Therefore,

$$\begin{aligned} (P'Q')^2 &= (PQ)^2 \\ 2 - 2\cos(A - B) &= 2 - 2(\cos A \cos B + \sin A \sin B) \\ -2\cos(A - B) &= -2(\cos A \cos B + \sin A \sin B) \end{aligned}$$

$$\mathbf{\cos(A - B) = \cos A \cos B + \sin A \sin B}$$

At the beginning of this section we showed that:

$$\cos(90° - 60°) \neq \cos 90° - \cos 60°$$

Does the identity that we proved make it possible to find $\cos(90° - 60°)$? We can check.

$$\begin{aligned} \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \cos(90° - 60°) &\stackrel{?}{=} \cos 90° \cos 60° + \sin 90° \sin 60° \\ \cos 30° &\stackrel{?}{=} 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{\sqrt{3}}{2}\right) \\ \tfrac{\sqrt{3}}{2} &\stackrel{?}{=} 0 + \tfrac{\sqrt{3}}{2} \\ \tfrac{\sqrt{3}}{2} &= \tfrac{\sqrt{3}}{2} \checkmark \end{aligned}$$

EXAMPLE 1

Use $(60° - 45°) = 15°$ to find the exact value of $\cos 15°$.

Solution

$$\begin{aligned}
\cos (A - B) &= \cos A \cos B + \sin A \sin B \\
\cos (60° - 45°) &= \cos 60° \cos 45° + \sin 60° \sin 45° \\
\cos 15° &= \left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) \\
\cos 15° &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
\cos 15° &= \frac{\sqrt{2} + \sqrt{6}}{4} \quad \textit{Answer}
\end{aligned}$$

Note: Example 1 shows that $\cos 15°$ is an *irrational* number.

Cosine of $(90° - B)$

The cofunction relationship between cosine and sine can be proved using $\cos (A - B)$. Use the identity for $\cos (A - B)$ to express $\cos (90° - B)$ in terms of a function of B.

Let $A = 90°$.

$$\begin{aligned}
\cos (A - B) &= \cos A \cos B + \sin A + \sin B \\
\cos (90° - B) &= \cos 90° \cos B + \sin 90° \sin B \\
\cos (90° - B) &= 0 \cos B + 1 \sin B \\
\cos (90° - B) &= \sin B
\end{aligned}$$

This is an identity, a statement true for all values of B.

EXAMPLE 2

Use the identity $\cos (90° - B) = \sin B$ to find $\sin (90° - B)$.

Solution Let $B = (90° - A)$.

$$\begin{aligned}
\cos (90° - B) &= \sin B \\
\cos (90° - (90° - A)) &= \sin (90° - A) \\
\cos (90° - 90° + A) &= \sin (90° - A) \\
\cos A &= \sin (90° - A) \quad \textit{Answer}
\end{aligned}$$

EXAMPLE 3

Given that A and B are second-quadrant angles, $\sin A = \frac{1}{3}$, and $\sin B = \frac{1}{5}$, find $\cos (A - B)$.

Solution Use the identity $\cos^2 \theta = 1 - \sin^2 \theta$ to find $\cos A$ and $\cos B$.

$$\cos^2 A = 1 - \sin^2 A$$
$$\cos^2 A = 1 - \left(\tfrac{1}{3}\right)^2$$
$$\cos^2 A = 1 - \tfrac{1}{9}$$
$$\cos^2 A = \tfrac{8}{9}$$
$$\cos A = \pm\sqrt{\tfrac{8}{9}}$$

A is in the second quadrant.

$$\cos A = -\frac{2\sqrt{2}}{3}$$

$$\cos^2 B = 1 - \sin^2 B$$
$$\cos^2 B = 1 - \left(\tfrac{1}{5}\right)^2$$
$$\cos^2 B = 1 - \tfrac{1}{25}$$
$$\cos^2 B = \tfrac{24}{25}$$
$$\cos B = \pm\sqrt{\tfrac{24}{25}}$$

B is in the second quadrant.

$$\cos B = -\frac{2\sqrt{6}}{5}$$

Use the identity for the cosine of the difference of two angles.

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\cos (A - B) = \left(-\frac{2\sqrt{2}}{3}\right)\left(-\frac{2\sqrt{6}}{5}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{5}\right)$$
$$\cos (A - B) = \frac{4\sqrt{12}}{15} + \frac{1}{15}$$
$$\cos (A - B) = \frac{4\sqrt{12} + 1}{15}$$
$$\cos (A - B) = \frac{8\sqrt{3} + 1}{15}$$

Note that since $\cos (A - B)$ is positive, $(A - B)$ must be a first- or fourth-quadrant angle.

Answer $\cos (A - B) = \dfrac{8\sqrt{3} + 1}{15}$

SUMMARY

We have proved the following identities:

$$\mathbf{\cos (A - B) = \cos A \cos B + \sin A \sin B}$$

$$\mathbf{\cos (90° - B) = \sin B}$$

$$\mathbf{\sin (90° - A) = \cos A}$$

Exercises

Writing About Mathematics

1. Are the equations $\sin \theta = \cos (90° - \theta)$ and $\cos \theta = \sin (90° - \theta)$ true for all real numbers or only for values of θ in the interval $0 < \theta < 90°$?

2. Emily said that, without finding the values on a calculator, she knows that $\sin 100° = \cos (-10°)$. Do you agree with Emily? Explain why or why not.

Developing Skills

In 3–17, find the exact value of cos $(A - B)$ for each given pair of values.

3. $A = 180°, B = 60°$
4. $A = 180°, B = 45°$
5. $A = 180°, B = 30°$
6. $A = 270°, B = 60°$
7. $A = 270°, B = 30°$
8. $A = 60°, B = 90°$
9. $A = 30°, B = 90°$
10. $A = 90°, B = 60°$
11. $A = 60°, B = 270°$
12. $A = 45°, B = 270°$
13. $A = 30°, B = 270°$
14. $A = 360°, B = 60°$
15. $A = \pi, B = \frac{2\pi}{3}$
16. $A = \frac{\pi}{6}, B = \frac{4\pi}{3}$
17. $A = \frac{3\pi}{4}, B = \frac{7\pi}{4}$

Applying Skills

In 18–20, show all work.

18. a. Find the exact value of cos 15° by using cos (45° − 30°).

b. Use the value of cos 15° found in **a** to find cos 165° by using cos (180° − 15°).

c. Use the value of cos 15° found in **a** to find cos 345° by using cos (360° − 15°).

d. Use cos A = sin $(90° - A)$ to find the exact value of sin 75°.

19. a. Find the exact value of cos 120° by using cos (180° − 60°).

b. Find the exact value of sin 120° by using $\cos^2 \theta + \sin^2 \theta = 1$ and the value of cos 120° found in **a**.

c. Find the exact value of cos 75° by using cos (120° − 45°).

d. Use the value of cos 75° found in **c** to find cos 105° by using cos (180° − 75°).

e. Use the value of cos 75° found in **c** to find cos 285° by using cos (360° − 75°).

f. Find the exact value of sin 15°.

20. a. Find the exact value of cos 210° by using cos (270° − 60°).

b. Find the exact value of sin 210° by using $\cos^2 \theta + \sin^2 \theta = 1$ and the value of cos 210° found in **a**.

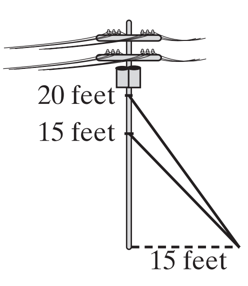
c. Find the exact value of cos 165° by using cos (210° − 45°).

d. Use the value of cos 165° found in **c** to find cos (−15°) by using cos (165° − 180°).

e. Use the value of cos (−15°) found in **d** to find cos 195° by using cos (180° − (−15°)).

f. Use the value of cos (−15°) found in **d** to find the exact value of sin 105°.

21. A telephone pole is braced by two wires that are both fastened to the ground at a point 15 feet from the base of the pole. The shorter wire is fastened to the pole 15 feet above the ground and the longer wire 20 feet above the ground.



a. What is the measure, in degrees, of the angle that the shorter wire makes with the ground?

b. Let θ be the measure of the angle that the longer wire makes with the ground. Find $\sin \theta$ and $\cos \theta$.

c. Find the cosine of the angle between the wires where they meet at the ground.

d. Find, to the nearest degree, the measure of the angle between the wires.

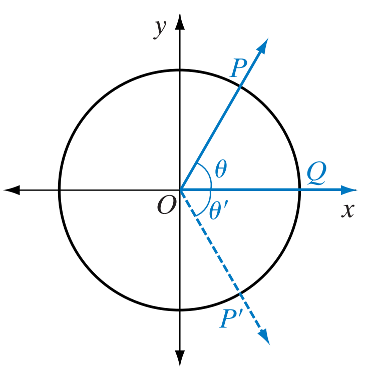
12-4 COSINE $(A + B)$

We know that $\cos(A + B)$ can be written as $\cos(A - (-B))$. Therefore,

$$\cos(A + B) = \cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$

We would like to write this identity in term of $\cos B$ and $\sin B$. Therefore, we want to find the relationship between $\cos B$ and $\cos(-B)$ and between $\sin A$ and $\sin(-A)$. In the exercises of Sections 11-1 and 11-2, we showed that $y = \sin x$ is an odd function and $y = \cos x$ is an even function. That is, for all values of x, $-\sin x = \sin(-x)$ and $\cos x = \cos(-x)$. We can establish these results graphically on the unit circle as the following hands-on activity demonstrates.

Hands-On Activity



1. Draw a first-quadrant angle in standard position. Let the point of intersection of the initial side with the unit circle be Q and the intersection of the terminal side with the unit circle be P. Let the measure of the angle be θ; $m\angle QOP = \theta$.

2. Reflect $\angle QOP$ in the x-axis. The image of $\angle QOP$ is $\angle QOP'$ and $m\angle QOP' = -\theta$.

3. Express $\sin \theta$, $\cos \theta$, $\sin(-\theta)$, and $\cos(-\theta)$ as the lengths of line segments. Show that $\cos(-\theta) = \cos \theta$ and that $\sin(-\theta) = -\sin \theta$.

4. Repeat steps 1 through 3 for a second-quadrant angle.

5. Repeat steps 1 through 3 for a third-quadrant angle.

6. Repeat steps 1 through 3 for a fourth-quadrant angle.

An algebraic proof can be used to prove the relationships $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$.

Proof of $\cos(-\theta) = \cos\theta$

In the identity for $\cos(A - B)$, let $A = 0$ and $B = \theta$.

$$\begin{aligned}
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\cos(0 - \theta) &= \cos 0 \cos\theta + \sin 0 \sin\theta \\
\cos(-\theta) &= 1 \cos\theta + 0 \sin\theta \\
\cos(-\theta) &= \cos\theta + 0 \\
\cos(-\theta) &= \cos\theta \checkmark
\end{aligned}$$

Proof of $\sin(-\theta) = -\sin\theta$

In the identity $\sin A = \cos(90° - A)$, let $A = -\theta$.

$$\begin{aligned}
\sin(-\theta) &= \cos(90° - (-\theta)) \\
\sin(-\theta) &= \cos(90° + \theta) \\
\sin(-\theta) &= \cos(\theta + 90°) \\
\sin(-\theta) &= \cos(\theta - (-90°)) \\
\sin(-\theta) &= \cos\theta \cos(-90°) + \sin\theta \sin(-90°) \\
\sin(-\theta) &= (\cos\theta)(0) + (\sin\theta)(-1) \\
\sin(-\theta) &= 0 - \sin\theta \\
\sin(-\theta) &= -\sin\theta \checkmark
\end{aligned}$$

Proof of the identity for $\cos(A + B)$

$$\begin{aligned}
\cos(A + B) &= \cos(A - (-B)) \\
\cos(A + B) &= \cos A \cos(-B) + \sin A \sin(-B) \\
\cos(A + B) &= \cos A \cos B + \sin A (-\sin B)
\end{aligned}$$

$$\mathbf{\cos(A + B) = \cos A \cos B - \sin A \sin B}$$

EXAMPLE 1

Find the exact value of $\cos 105° = \cos(45° + 60°)$ using identities.

Solution

$$\begin{aligned}
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos(45° + 60°) &= \cos 45° \cos 60° - \sin 45° \sin 60° \\
\cos 105° &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\
\cos 105° &= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} \\
\cos 105° &= \frac{\sqrt{2} - \sqrt{6}}{4} \quad \textit{Answer}
\end{aligned}$$

EXAMPLE 2

Show that $\cos(\pi + \theta) = -\cos\theta$.

Solution We are now working in radians.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(\pi + \theta) = \cos\pi\cos\theta - \sin\pi\sin\theta$$
$$\cos(\pi + \theta) = -1\cos\theta - 0\sin\theta$$
$$\cos(\pi + \theta) = -\cos\theta \checkmark$$

SUMMARY

We have proved the following identities:

$$\boldsymbol{\cos(-\theta) = \cos\theta}$$

$$\boldsymbol{\sin(-\theta) = -\sin\theta}$$

$$\boldsymbol{\cos(A + B) = \cos A\cos B - \sin A\sin B}$$

Exercises

Writing About Mathematics

1. Maggie said that $\cos(A + B) + \cos(A - B) = \cos 2A$. Do you agree with Maggie? Justify your answer.

2. Germaine said $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$. Do you agree with Germaine? Justify your answer.

Developing Skills

In 3–17, find the exact value of $\cos(A + B)$ for each given pair of values.

3. $A = 90°, B = 60°$
4. $A = 90°, B = 45°$
5. $A = 90°, B = 30°$
6. $A = 180°, B = 60°$
7. $A = 180°, B = 30°$
8. $A = 180°, B = 45°$
9. $A = 270°, B = 30°$
10. $A = 270°, B = 60°$
11. $A = 270°, B = 45°$
12. $A = 60°, B = 60°$
13. $A = 60°, B = 270°$
14. $A = 45°, B = 270°$
15. $A = \frac{\pi}{2}, B = \frac{2\pi}{3}$
16. $A = \frac{\pi}{3}, B = \frac{4\pi}{3}$
17. $A = \frac{\pi}{4}, B = \frac{\pi}{6}$

Applying Skills

In 18–20, show all work.

18. a. Find the exact value of cos 75° by using cos (45° + 30°).

b. Use the value of cos 75° found in **a** to find cos 255° by using cos (180° + 75°).

c. Use cos A = sin (90° − A) to find the exact value of sin 15°.

19. a. Find the exact value of cos 120° by using cos (60° + 60°).

b. Find the exact value of sin 120° by using $\cos^2 \theta + \sin^2 \theta = 1$ and the value of cos 120° found in **a**.

c. Find the exact value of cos 165° by using cos (120° + 45°).

d. Use the value of cos 165° found in **c** to find cos 345° by using cos (180° + 165°).

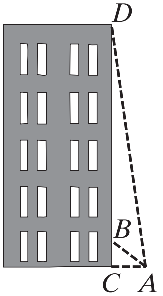
20. a. Find the exact value of cos 315° by using cos (270° + 45°).

b. Find the exact value of sin 315° by using $\cos^2 \theta + \sin^2 \theta = 1$ and the value of cos 315° found in **a**.

c. Find the exact value of cos 345° by using cos (315° + 30°).

d. Explain why cos 405° = cos 45°.

21. An engineer wants to determine CD, the exact height of a building. To do this, he first locates B on $\overline{CD}$, a point 30 feet above C at the foot of the building. Then he locates A, a point on the ground 40 feet from C. From A, the engineer then finds that the angle of elevation of D is 45° larger than θ, the angle of elevation of B.



a. Find AB, sin θ, and cos θ.

b. Use sin θ and cos θ found in **a** to find the exact value of cos (θ + 45°).

c. Use the value of cos (θ + 45°) found in **b** to find AD.

d. Find CD, the height of the building.

12-5 SINE (A − B) AND SINE (A + B)

We can use the cofunction identity sin θ = cos (90° − θ) and the identities for cos (A − B) and cos (A + B) to derive identities for sin (A − B) and sin (A + B).

Sine of (A − B)

Let $\theta = A - B$.

$$\begin{aligned}
\sin\theta &= \cos(90^\circ - \theta)\\
\sin(A - B) &= \cos(90^\circ - (A - B))\\
&= \cos(90^\circ - A + B)\\
&= \cos((90^\circ - A) + B)\\
&= \cos(90^\circ - A)\cos B - \sin(90^\circ - A)\sin B\\
&= \sin A\cos B - \cos A\sin B
\end{aligned}$$

$$\mathbf{\sin(A - B) = \sin A\cos B - \cos A\sin B}$$

Sine of (A + B)

The derivation of an identity for $\sin(A + B)$ is similar.

Let $\theta = A + B$.

$$\begin{aligned}
\sin\theta &= \cos(90^\circ - \theta)\\
\sin(A + B) &= \cos(90^\circ - (A + B))\\
&= \cos(90^\circ - A - B)\\
&= \cos((90^\circ - A) - B)\\
&= \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B\\
&= \sin A\cos B + \cos A\sin B
\end{aligned}$$

$$\mathbf{\sin(A + B) = \sin A\cos B + \cos A\sin B}$$

EXAMPLE 1

Use $\sin(45^\circ - 30^\circ)$ to find the exact value of $\sin 15^\circ$.

Solution

$$\begin{aligned}
\sin(A - B) &= \sin A\cos B - \cos A\sin B\\
\sin(45^\circ - 30^\circ) &= \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ\\
\sin 15^\circ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\
&= \frac{\sqrt{6} - \sqrt{2}}{4} \quad \textit{Answer}
\end{aligned}$$

EXAMPLE 2

Use sin (60° + 45°) to find the exact value of sin 105°.

Solution

$$\begin{aligned}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\sin(60° + 45°) &= \sin 60° \cos 45° + \cos 60° \sin 45° \\
\sin 105° &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6} + \sqrt{2}}{4} \quad \textit{Answer}
\end{aligned}$$

EXAMPLE 3

Show that $\sin(\pi + \theta) = -\sin\theta$.

Solution We are now working in radians.

$$\begin{aligned}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\sin(\pi + \theta) &= \sin \pi \cos \theta + \cos \pi \sin \theta \\
\sin(\pi + \theta) &= 0 \cos \theta + (-1) \sin \theta \\
\sin(\pi + \theta) &= 0 - \sin \theta \\
\sin(\pi + \theta) &= -\sin \theta \checkmark
\end{aligned}$$

SUMMARY

We have proved the following identities:

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Exercises

Writing About Mathematics

1. William said that $\sin(A + B) + \sin(A - B) = \sin 2A$. Do you agree with William? Justify your answer.

2. Freddy said that $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$. Do you agree with Freddy? Justify your answer.

Developing Skills

In 3–17, find the exact value of sin $(A - B)$ and of sin $(A + B)$ for each given pair of values.

3. $A = 180°, B = 60°$
4. $A = 180°, B = 45°$
5. $A = 180°, B = 30°$
6. $A = 270°, B = 60°$
7. $A = 270°, B = 30°$
8. $A = 60°, B = 90°$
9. $A = 30°, B = 90°$
10. $A = 90°, B = 60°$
11. $A = 60°, B = 270°$
12. $A = 45°, B = 270°$
13. $A = 30°, B = 270°$
14. $A = 360°, B = 60°$
15. $A = \frac{3\pi}{2}, B = 2\pi$
16. $A = \frac{2\pi}{3}, B = \frac{\pi}{6}$
17. $A = \frac{\pi}{3}, B = \frac{5\pi}{4}$

Applying Skills

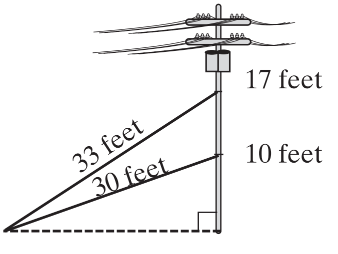
In 18–20, show all work.

18. a. Find the exact value of sin 15° by using sin (45° − 30°).
b. Use the value of sin 15° found in **a** to find sin 165° by using sin (180° − 15°).
c. Use the value of sin 15° found in **a** to find sin 345° by using sin (360° − 15°).
d. Use the value of sin 15° found in **a** to find sin 195° by using sin (180° + 15°).

19. a. Find the exact value of sin 120° by using sin (180° − 60°).
b. Find the exact value of cos 120° by using cos (180° − 60°).
c. Find the exact value of sin 75° by using sin (120° − 45°).
d. Use the value of sin 75° found in **c** to find sin 105° by using sin (180° − 75°).
e. Use the value of sin 75° found in **c** to find sin 285° by using sin (360° − 75°).

20. a. Find the exact value of sin 210° by using sin (270° − 60°).
b. Find the exact value of cos 210° by using cos (270° − 60°).
c. Find the exact value of sin 165° by using sin (210° − 45°).
d. Use the value of sin 165° found in **c** to find sin (−15°) by using sin (165° − 180°).
e. Use the value of sin (−15°) found in **d** to find sin 195° by using sin (180° − (−15°)).
f. Use the value of sin (−15°) found in **d** to find the exact value of sin 105°.

21. A telephone pole is braced by two wires that are both fastened to the ground at the same point. The shorter wire is 30 feet long and is fastened to the pole 10 feet above the foot of the pole. The longer wire is 33 feet long and is fastened to the pole 17 feet above the foot of the pole.



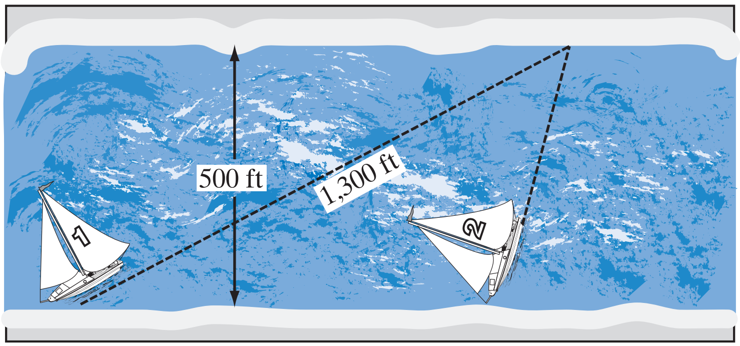
a. If the measure of the angle that the longer wire makes with the ground is x and the measure of the angle that the shorter wire makes with the ground is y, find sin x, cos x, sin y, and cos y.
b. Find the exact value of sin $(x - y)$, the sine of the angle between the two wires.
c. Find to the nearest degree the measure of the angle between the two wires.

22. Two boats leave the same dock to cross a river that is 500 feet wide. The first boat leaves the dock at an angle of θ with the shore and travels 1,300 feet to reach a point downstream on the opposite shore of the river. The second boat leaves the dock at an angle of $(\theta + 30°)$ with the shore.

a. Find $\sin \theta$ and $\cos \theta$.

b. Find $\sin (\theta + 30°)$.

c. Find the distance that the second boat travels to reach the opposite shore.



23. The coordinates of any point in the coordinate plane can be written as $A(r \cos a, r \sin a)$ when a is the measure of the angle between the positive ray of the x-axis and $\overrightarrow{OA}$ and $r = OA$. Under a rotation of θ about the origin, the coordinates of A', the image of A, are $(r \cos (a + \theta), r \sin (a + \theta))$. Find the coordinates of A', the image of $A(6, 8)$, under a rotation of 45° about the origin.

12-6 TANGENT ($A - B$) AND TANGENT ($A + B$)

We can use the identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and the identities for $\sin (A + B)$ and $\cos (A + B)$ to write identities for $\tan (A + B)$ and $\tan (A - B)$.

Tangent of ($A + B$)

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

We would like to write this identity for $\tan (A + B)$ in terms of $\tan A$ and $\tan B$. We can do this by dividing each term of the numerator and each term of the denominator by $\cos A \cos B$. When we do this we are dividing by a fraction equal to 1 and therefore leaving the value of the expression unchanged.

$$\tan (A + B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\frac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \frac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\frac{\cancel{\cos A} \cancel{\cos B}}{\cancel{\cos A} \cancel{\cos B}} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\boldsymbol{\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

Tangent of $(A - B)$

The identity for tan $(A - B)$ can be derived in a similar manner.

$$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

$$\tan(A - B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\frac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} - \frac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\frac{\cancel{\cos A \cos B}}{\cancel{\cos A \cos B}} + \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\mathbf{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

These identities are true for all replacements of A and B for which $\cos A \neq 0$ and $\cos B \neq 0$, and for which $\tan(A + B)$ or $\tan(A - B)$ are defined.

EXAMPLE 1

Use $\tan 2\pi = 0$ and $\tan \frac{\pi}{4} = 1$ to show that $\tan \frac{7\pi}{4} = -1$.

Solution

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan\left(2\pi - \frac{\pi}{4}\right) = \frac{\tan 2\pi - \tan \frac{\pi}{4}}{1 + \tan 2\pi \tan \frac{\pi}{4}}$$

$$\tan \frac{7\pi}{4} = \frac{0 - 1}{1 + (0)(1)}$$

$$\tan \frac{7\pi}{4} = -1 \checkmark$$

EXAMPLE 2

Use $(45° + 120°) = 165°$ to find the exact value of tan 165°.

Solution $\sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$, so $\tan 45° = 1$.

$\sin 120° = \frac{\sqrt{3}}{2}$ and $\cos 120° = -\frac{1}{2}$, so $\tan 120° = -\sqrt{3}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(45° + 120°) = \frac{\tan 45° + \tan 120°}{1 - \tan 45° \tan 120°}$$

$$\tan 165° = \frac{1 + (-\sqrt{3})}{1 - (1)(-\sqrt{3})}$$

$$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$

$$\tan 165° = \frac{4 - 2\sqrt{3}}{-2} = -2 + \sqrt{3}$$

Alternative Solution Write the identity $\tan(A + B)$ in terms of sine and cosine.

$$\tan(A + B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$\tan(45° + 120°) = \frac{\sin 45° \cos 120° + \cos 45° \sin 120°}{\cos 45° \cos 120° - \sin 45° \sin 120°}$$

$$\tan 165° = \frac{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{-\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}}{-\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}} \times \frac{\frac{4}{\sqrt{2}}}{\frac{4}{\sqrt{2}}}$$

$$\tan 165° = \frac{-1 + \sqrt{3}}{-1 - \sqrt{3}} = -2 + \sqrt{3}$$

Answer $\tan 165° = -2 + \sqrt{3}$ or $\sqrt{3} - 2$

SUMMARY

We have proved the following identities:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Exercises

Writing About Mathematics

1. Explain why the identity $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ is not valid when A or B is equal to $\frac{\pi}{2} + n\pi$ for any integer n.

2. Explain why $\frac{\tan A + \tan B}{1 - \tan A \tan B}$ is undefined when $A = \frac{\pi}{6}$ and $B = \frac{\pi}{3}$.

Developing Skills

In 3–17, find the exact value of $\tan(A + B)$ and of $\tan(A - B)$ for each given pair of values.

3. $A = 45°, B = 30°$ **4.** $A = 45°, B = 60°$ **5.** $A = 60°, B = 60°$

6. $A = 180°, B = 30°$ **7.** $A = 180°, B = 45°$ **8.** $A = 180°, B = 60°$

9. $A = 120°, B = 30°$ **10.** $A = 120°, B = 45°$ **11.** $A = 120°, B = 60°$

12. $A = 120°, B = 120°$ **13.** $A = 240°, B = 120°$ **14.** $A = 360°, B = 60°$

15. $A = \pi, B = \frac{\pi}{3}$ **16.** $A = \frac{5\pi}{6}, B = \frac{5\pi}{6}$ **17.** $A = \frac{\pi}{3}, B = \frac{\pi}{4}$

Applying Skills

18. Prove that $\tan(180° + \theta) = \tan\theta$.

19. Find $\tan(A + B)$ if $\tan A = 3$ and $\tan B = -\frac{1}{2}$.

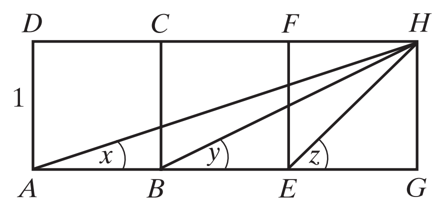
20. Find $\tan(A - B)$ if $\tan A = \frac{3}{4}$ and $\tan B = -8$.

21. Find $\tan(A + B)$ if A is in the second quadrant, $\sin A = 0.6$, and $\tan B = 4$.

22. If $A = \arctan 2$ and $B = \arctan(-2)$, find $\tan(A - B)$.

23. If $A = \arctan\left(-\frac{2}{3}\right)$ and $B = \arctan\frac{2}{3}$, find $\tan(A + B)$.

24. In the diagram, $ABCD$, $BEFC$, and $EGHF$ are congruent squares with $AD = 1$. Let $m\angle GAH = x$, $m\angle GBH = y$, and $m\angle GEH = z$.



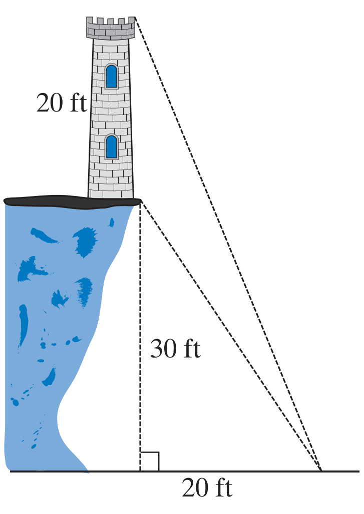
a. Find $\tan(x + y)$.

b. Does $x + y = z$? Justify your answer.

25. A tower that is 20 feet tall stands at the edge of a 30-foot cliff. From a point on level ground that is 20 feet from a point directly below the tower at the base of the cliff, the measure of the angle of elevation of the top of the tower is x and the measure of the angle of elevation of the foot of the tower is y.

a. Find the exact value of $\tan(x - y)$, the tangent of the angle between the lines of sight to the foot and top of the tower.

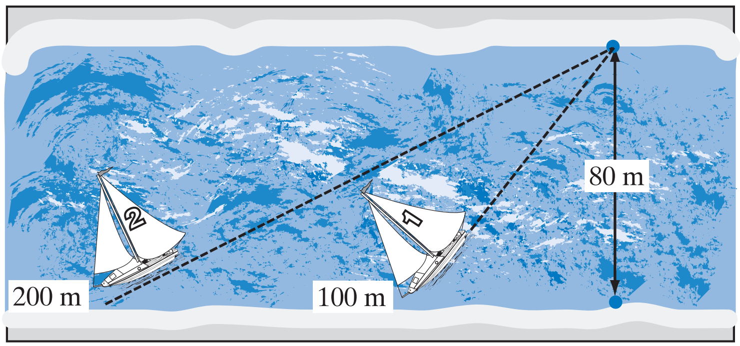
b. Find to the nearest degree the measure of the angle between the lines of sight to the foot and the top of the tower.



26. Two boats leave a dock to cross a river that is 80 meters wide. The first boat travels to a point that is 100 meters downstream from a point directly opposite the starting point, and the second boat travels to a point that is 200 meters downstream from a point directly opposite the starting point.

a. Let x be the measure of the angle between the river's edge and the path of the first boat and y be the measure of the angle between the river's edge and the path of the second boat. Find $\tan x$ and $\tan y$.

b. Find the tangent of the measure of the angle between the paths of the boats.



12-7 FUNCTIONS OF 2*A*

The identities for $\sin(A + B)$, $\cos(A + B)$, and $\tan(A + B)$ are true when $A = B$. These identities can be used to find the function values of $2A$. We often call the identities used to find the function values of twice an angle **double-angle formulas**.

Sine of 2*A*

In the identity for $\sin(A + B)$, let $B = A$.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$
$$\mathbf{\sin(2A) = 2 \sin A \cos A}$$

Cosine of 2*A*

In the identity for $\cos(A + B)$, let $B = A$.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$
$$\mathbf{\cos(2A) = \cos^2 A - \sin^2 A}$$

The identity for cos 2*A* can be written in two other ways:

$$\cos(2A) = 2\cos^2 A - 1$$

$$\cos(2A) = 1 - 2\sin^2 A$$

Proof of the first of these identities are given in Example 2. The proof of the second identity is left to the student in Exercise 23.

Tangent of 2A

In the identity for $\tan(A+B)$, let $B = A$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

EXAMPLE 1

If $\tan\theta = -\frac{\sqrt{7}}{3}$ and θ is a second-quadrant angle, find:

a. $\sec\theta$ **b.** $\cos\theta$ **c.** $\sin\theta$

d. $\sin 2\theta$ **e.** $\cos 2\theta$ **f.** $\tan 2\theta$

g. In what quadrant does 2θ lie?

Solution **a.** Use the identity $1 + \tan^2\theta = \sec^2\theta$ to find $\sec\theta$. Since θ is a second-quadrant angle, $\sec\theta$ is negative.

$$1 + \left(-\frac{\sqrt{7}}{3}\right)^2 = \sec^2\theta$$

$$1 + \frac{7}{9} = \sec^2\theta$$

$$\frac{16}{9} = \sec^2\theta$$

$$-\frac{4}{3} = \sec\theta \quad \textit{Answer}$$

b. Use $\sec\theta = \frac{1}{\cos\theta}$ to find $\cos\theta$.

$$-\frac{4}{3} = \frac{1}{\cos\theta}$$

$$-\frac{3}{4} = \cos\theta \quad \textit{Answer}$$

c. Use $\cos^2\theta + \sin^2\theta = 1$ to find $\sin\theta$. Since θ is a second-quadrant angle, $\sin\theta$ will be positive.

$$\left(-\frac{3}{4}\right)^2 + \sin^2\theta = 1$$

$$\frac{9}{16} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{7}{16}$$

$$\sin\theta = \frac{\sqrt{7}}{4} \quad \textit{Answer}$$

d. $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\left(\frac{\sqrt{7}}{4}\right)\left(-\frac{3}{4}\right)$$

$$= -\frac{3\sqrt{7}}{8} \quad \textit{Answer}$$

e. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{3}{4}\right)^2 - \left(\frac{\sqrt{7}}{4}\right)^2$$

$$= \frac{9}{16} - \frac{7}{16}$$

$$= \frac{2}{16} = \frac{1}{8} \quad \textit{Answer}$$

f. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$$= \frac{2\left(-\frac{\sqrt{7}}{3}\right)}{1 - \left(-\frac{\sqrt{7}}{3}\right)^2}$$

$$= \frac{-\frac{2\sqrt{7}}{3}}{\frac{2}{9}} = -\frac{6\sqrt{7}}{2} = -3\sqrt{7} \quad \textit{Answer}$$

g. Since $\sin 2\theta$ and $\tan 2\theta$ are negative and $\cos 2\theta$ is positive, 2θ must be a fourth-quadrant angle. *Answer*

EXAMPLE 2

Prove the identity $\cos 2\theta = 2\cos^2\theta - 1$.

Solution First write the identity for $\cos 2\theta$ that we have already proved. Then substitute for $\sin^2\theta$ using the Pythagorean identity $\sin^2\theta = 1 - \cos^2\theta$.

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= \cos^2\theta - 1 + \cos^2\theta \\ &= 2\cos^2\theta - 1 \ ✔\end{aligned}$$

Alternative Solution To the right side of the identity for $\cos 2\theta$, add 0 in the form $\cos^2\theta - \cos^2\theta$.

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - \sin^2\theta + \cos^2\theta - \cos^2\theta \\ &= \cos^2\theta + \cos^2\theta - (\sin^2\theta + \cos^2\theta) \\ &= 2\cos^2\theta - 1 \ ✔\end{aligned}$$

SUMMARY

We have proved the following identities:

$$\mathbf{\sin 2A = 2\sin A\cos A}$$

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$$

$$\mathbf{\cos 2A = 2\cos^2 A - 1}$$

$$\mathbf{\cos 2A = 1 - 2\sin^2 A}$$

$$\mathbf{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Exercises

Writing About Mathematics

1. Does $\cos 2\theta = \sin 2(90° - \theta)$? Justify your answer.

2. Does $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$? Justify your answer.

Developing Skills

In 3–8, for each value of θ, use double-angle formulas to find **a.** $\sin 2\theta$, **b.** $\cos 2\theta$, **c.** $\tan 2\theta$. Show all work.

3. $\theta = 30°$
4. $\theta = 225°$
5. $\theta = 330°$
6. $\theta = \frac{\pi}{4}$
7. $\theta = \frac{7\pi}{6}$
8. $\theta = \frac{5\pi}{3}$

In 9–20, for each given function value, find **a.** $\sin 2\theta$, **b.** $\cos 2\theta$, **c.** $\tan 2\theta$, **d.** the quadrant in which 2θ lies. Show all work.

9. $\tan \theta = \frac{3}{5}$, θ in the first quadrant
10. $\tan \theta = -\frac{\sqrt{11}}{5}$ in the second quadrant
11. $\sin \theta = \frac{2\sqrt{6}}{7}$ in the first quadrant
12. $\sin \theta = -0.5$ in the third quadrant
13. $\cos \theta = \frac{2\sqrt{10}}{7}$ in the first quadrant
14. $\cos \theta = -\frac{2\sqrt{5}}{5}$ in the second quadrant
15. $\tan \theta = \frac{12}{5}$ in the third quadrant
16. $\sin \theta = -\frac{\sqrt{2}}{3}$ in the third quadrant
17. $\csc \theta = \sqrt{5}$ in the second quadrant
18. $\sec \theta = \frac{\sqrt{13}}{2}$ in the fourth quadrant
19. $\cot \theta = -\frac{1}{3}$ in the second quadrant
20. $\tan \theta = 2$ in the third quadrant

Applying Skills

21. Prove the identity: $\cot \theta = \frac{\sin 2\theta}{1 - \cos 2\theta}$.

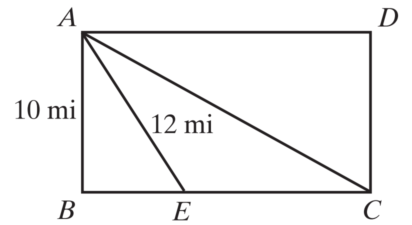
22. Prove the identity: $\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta} = \csc \theta$.

23. Show that $\cos 2\theta = 1 - 2\sin^2 \theta$.

24. Show that $\csc 2\theta = \frac{1}{2} \sec \theta \csc \theta$.

25. **a.** Derive an identity for $\sin 4A$ in terms of the functions of $2A$.
    b. Derive an identity for $\cos 4A$ in terms of the functions of $2A$.
    c. Derive an identity for $\tan 4A$ in terms of the functions of $2A$.
    Hint: Let $4A = 2\theta$.

26. A park in the shape of a rectangle, $ABCD$, is crossed by two paths: $\overline{AC}$, a diagonal, and $\overline{AE}$, which intersects $\overline{BC}$ at E. The measure of $\angle BAC$ is twice the measure of $\angle BAE$, $AB = 10$ miles and $AE = 12$ miles.



a. Let $m\angle BAE = \theta$. Find the exact value of $\cos\theta$.

b. Express $m\angle BAC$ in terms of θ and find the exact value of the cosine of $\angle BAC$.

c. Find the exact value of AC.

27. The image of $A(4, 0)$ under a rotation of θ about the origin is $A'(\sqrt{10}, \sqrt{6})$. What are the coordinates of A'', the image of A' under the same rotation?

12-8 FUNCTIONS OF $\frac{1}{2}A$

Just as there are identities to find the function values of $2A$, there are identities to find $\cos\frac{1}{2}A$, $\sin\frac{1}{2}A$, and $\tan\frac{1}{2}A$. We often call the identities used to find the function values of half an angle **half-angle formulas**.

Cosine of $\frac{1}{2}A$

We can use the identity for $\cos 2\theta$ to write an identity for $\frac{1}{2}A$. Begin with the identity for $\cos 2\theta$ written in terms of $\cos\theta$. Then solve for $\cos\theta$.

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 \\
1 + \cos 2\theta &= 2\cos^2\theta \\
\frac{1 + \cos 2\theta}{2} &= \cos^2\theta \\
\pm\sqrt{\frac{1 + \cos 2\theta}{2}} &= \cos\theta \\
\cos\theta &= \pm\sqrt{\frac{1 + \cos 2\theta}{2}}
\end{aligned}$$

Let $2\theta = A$ and $\theta = \frac{1}{2}A$.

$$\cos\frac{1}{2}A = \pm\sqrt{\frac{1 + \cos A}{2}}$$

Sine of $\frac{1}{2}A$

Begin with the identity for $\cos 2\theta$, this time written in terms of $\sin\theta$. Then solve for $\sin\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$2\sin^2\theta = 1 - \cos 2\theta$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$
$$\sin\theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}$$

Let $2\theta = A$ and $\theta = \frac{1}{2}A$.

$$\sin\tfrac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{2}}$$

Tangent of $\frac{1}{2}A$

Use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Let $\theta = \frac{1}{2}A$. Then substitute in the values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$.

$$\tan\tfrac{1}{2}A = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}$$
$$\tan\tfrac{1}{2}A = \frac{\pm\sqrt{\frac{1 - \cos A}{2}}}{\pm\sqrt{\frac{1 + \cos A}{2}}}$$
$$\tan\tfrac{1}{2}A = \frac{\pm\sqrt{\frac{1 - \cos A}{2}}}{\pm\sqrt{\frac{1 + \cos A}{2}}} \times \frac{\frac{\sqrt{2}}{1}}{\frac{\sqrt{2}}{1}}$$
$$\tan\tfrac{1}{2}A = \pm\frac{\sqrt{1 - \cos A}}{\sqrt{1 + \cos A}}$$

When we use the identities for the function values of $(A + B)$, $(A - B)$, and $2A$, the sign of the function value is a result of the computation. When we use the identities for the function values of $\frac{1}{2}A$, the sign of the function value must be chosen according to the quadrant in which $\frac{1}{2}A$ lies.

For example, if A is a third-quadrant angle such that $180° < A < 270°$, then $90° < \frac{1}{2}A < 135°$. Therefore, $\frac{1}{2}A$ is a second-quadrant angle. The sine value of $\frac{1}{2}A$ is positive and the cosine and tangent values of $\frac{1}{2}A$ are negative.

If A is a third-quadrant angle and $540° < A < 630°$, then $270° < \frac{1}{2}A < 315°$. Therefore, $\frac{1}{2}A$ is a fourth-quadrant angle. The cosine value of $\frac{1}{2}A$ is positive and the sine and tangent values of $\frac{1}{2}A$ are negative.

EXAMPLE 1

If $180° < A < 270°$ and $\sin A = -\frac{\sqrt{5}}{3}$, find:

a. $\sin \frac{1}{2}A$ **b.** $\cos \frac{1}{2}A$ **c.** $\tan \frac{1}{2}A$

Solution The identities for the function values of $\frac{1}{2}A$ are given in terms of $\cos A$. Use $\cos A = \pm\sqrt{1 - \sin^2 A}$ to find $\cos A$. Since A is a third-quadrant angle, $\cos A$ is negative.

$$\cos A = -\sqrt{1 - \left(\frac{\sqrt{5}}{3}\right)^2} = -\sqrt{\frac{9}{9} - \frac{5}{9}} = -\sqrt{\frac{4}{9}} = -\frac{2}{3}$$

If $180° < A < 270°$, then $90° < \frac{1}{2}A < 135°$. Therefore, $\frac{1}{2}A$ is a second-quadrant angle: $\sin \frac{1}{2}A$ is positive and $\cos \frac{1}{2}A$ and $\tan \frac{1}{2}A$ are negative. Use the half-angle identities to find the function values of $\frac{1}{2}A$.

a.
$$\begin{aligned}\sin \tfrac{1}{2}A &= \sqrt{\frac{1 - \cos A}{2}}\\ &= \sqrt{\frac{1 - \left(-\frac{2}{3}\right)}{2}}\\ &= \sqrt{\frac{5}{6}}\\ &= \frac{\sqrt{30}}{6}\end{aligned}$$

b.
$$\begin{aligned}\cos \tfrac{1}{2}A &= -\sqrt{\frac{1 + \cos A}{2}}\\ &= -\sqrt{\frac{1 + \left(-\frac{2}{3}\right)}{2}}\\ &= -\sqrt{\frac{1}{6}}\\ &= -\frac{\sqrt{6}}{6}\end{aligned}$$

c.
$$\begin{aligned}\tan \tfrac{1}{2}A &= -\sqrt{\frac{1 - \cos A}{1 + \cos A}}\\ &= -\sqrt{\frac{1 - \left(-\frac{2}{3}\right)}{1 + \left(-\frac{2}{3}\right)}}\\ &= -\sqrt{\frac{5}{1}}\\ &= -\sqrt{5}\end{aligned}$$

Answers **a.** $\sin \frac{1}{2}A = \frac{\sqrt{30}}{6}$ **b.** $\cos \frac{1}{2}A = -\frac{\sqrt{6}}{6}$ **c.** $\tan \frac{1}{2}A = -\sqrt{5}$

EXAMPLE 2

Show that $\tan \frac{1}{2}A = \pm\frac{\sin A}{1 + \cos A}$.

Solution

$$\begin{aligned}\tan \tfrac{1}{2}A &= \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}\\ &= \pm\sqrt{\frac{1 - \cos A}{1 + \cos A} \times \frac{1 + \cos A}{1 + \cos A}}\\ &= \pm\sqrt{\frac{1 - \cos^2 A}{(1 + \cos A)^2}}\\ &= \pm\sqrt{\frac{\sin^2 A}{(1 + \cos A)^2}}\\ &= \pm\frac{\sin A}{1 + \cos A} \checkmark\end{aligned}$$

SUMMARY

We have proved the following identities:

$$\cos \tfrac{1}{2}A = \pm\sqrt{\frac{1 + \cos A}{2}}$$

$$\sin \tfrac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{2}}$$

$$\tan \tfrac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

Exercises

Writing About Mathematics

1. Karla said that if $\cos A$ is positive, then $-\frac{\pi}{2} < A < \frac{\pi}{2}$, $-\frac{\pi}{4} < \frac{1}{2}A < \frac{\pi}{4}$, and $\cos \frac{1}{2}A$ is positive. Do you agree with Karla? Explain why or why not.

2. In Example 1, can $\tan \frac{1}{2}A$ be found by using $\dfrac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A}$? Explain why or why not.

Developing Skills

In 3–8, for each value of θ, use half-angle formulas to find **a.** $\sin \frac{1}{2}\theta$ **b.** $\cos \frac{1}{2}\theta$ **c.** $\tan \frac{1}{2}\theta$. Show all work.

3. $\theta = 480°$ **4.** $\theta = 120°$ **5.** $\theta = 300°$

6. $\theta = 2\pi$ **7.** $\theta = \frac{7\pi}{2}$ **8.** $\theta = \frac{3\pi}{2}$

In 9–14, for each value of $\cos A$, find **a.** $\sin \frac{1}{2}A$ **b.** $\cos \frac{1}{2}A$ **c.** $\tan \frac{1}{2}A$. Show all work.

9. $\cos A = \frac{3}{4}$, $0° < A < 90°$ **10.** $\cos A = -\frac{5}{12}$, $90° < A < 180°$

11. $\cos A = -\frac{1}{9}$, $180° < A < 270°$ **12.** $\cos A = \frac{1}{8}$, $300° < A < 360°$

13. $\cos A = -\frac{1}{5}$, $450° < A < 540°$ **14.** $\cos A = \frac{7}{9}$, $360° < A < 450°$

15. If $\sin A = \frac{24}{25}$ and $90° < A < 180°$, find: **a.** $\sin \frac{1}{2}A$ **b.** $\cos \frac{1}{2}A$ **c.** $\tan \frac{1}{2}A$

16. If $\sin A = -\frac{4}{5}$ and $180° < A < 270°$, find: **a.** $\sin \frac{1}{2}A$ **b.** $\cos \frac{1}{2}A$ **c.** $\tan \frac{1}{2}A$

17. If $\sin A = -\frac{24}{25}$ and $540° < A < 630°$, find: **a.** $\sin \frac{1}{2}A$ **b.** $\cos \frac{1}{2}A$ **c.** $\tan \frac{1}{2}A$

18. If $\tan A = 3$ and $180° < A < 270°$, find: **a.** $\sin \frac{1}{2}A$ **b.** $\cos \frac{1}{2}A$ **c.** $\tan \frac{1}{2}A$

Applying Skills

19. Show that $\tan \frac{1}{2}A = \pm\frac{1 - \cos A}{\sin A}$.

20. Use $\cos A = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ to show that the exact value of $\tan \frac{\pi}{8} = \sqrt{2} - 1$.

21. Use $\cos A = \cos 30° = \frac{\sqrt{3}}{2}$ to show that the exact value of $\tan 15° = 2 - \sqrt{3}$.

22. Use $\cos A = \cos 30° = \frac{\sqrt{3}}{2}$ to show that the exact value of $\sin 15° = \frac{\sqrt{2 - \sqrt{3}}}{2}$.

23. a. Derive an identity for $\sin \frac{1}{4}A$ in terms of $\cos \frac{1}{2}A$.

b. Derive an identity for $\cos \frac{1}{4}A$ in terms of $\cos \frac{1}{2}A$.

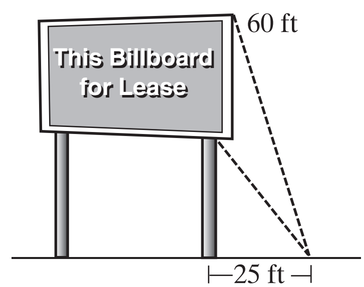
c. Derive an identity for $\tan \frac{1}{4}A$ in terms of $\cos \frac{1}{2}A$.

Hint: Let $\frac{1}{4}A = \frac{1}{2}\theta$.

24. The top of a billboard that is mounted on a base is 60 feet above the ground. At a point 25 feet from the foot of the base, the measure of the angle of elevation to the top of the base is one-half the measure of the angle of elevation to the top of the billboard.

a. Let θ be the measure of the angle of elevation to the top of the billboard. Find the $\cos \theta$ and $\tan \frac{1}{2}\theta$.

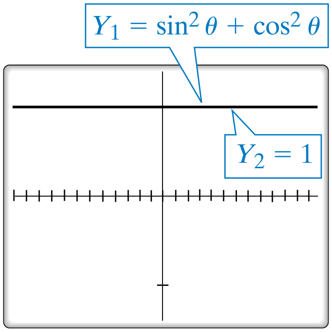
b. Find the height of the base and the height of the billboard.



Hands-On Activity: Graphical Support for the Trigonometric Identities

We can use the graphing calculator to provide support that an identity is true. Treat each side of the identity as a function and graph each function on the same set of axes. If the graphs of the functions coincide, then we have provided graphical support that the identity is true. Note that *support* is not the same as *proof*. In order to prove an identity, we need to use the algebraic methods from this chapter or similar algebraic methods.

For example, to provide support for the identity $\sin^2\theta + \cos^2\theta = 1$, treat each side as a function. In your graphing calculator, enter $Y_1 = \sin^2\theta + \cos^2\theta$ and $Y_2 = 1$. Graph both functions in the interval $-2\pi \le \theta \le 2\pi$. As the graph shows, both functions appear to coincide.



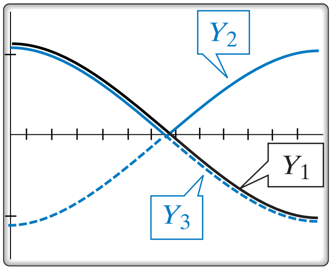
For 1–3, explore each equation on the graphing calculator. Do these equations appear to be identities?

1. $\cot\theta\sin 2\theta = 1 + \cos 2\theta$

2. $\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

3. $\tan^2\theta\cos^2\theta = 1 - \cos^2\theta$

4. Kevin tried to provide graphical support for the identity $\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}}$ by graphing the functions $Y_1 = \cos\left(\frac{X}{2}\right)$, $Y_2 = \sqrt{\frac{1+\cos X}{2}}$, $Y_3 = -\sqrt{\frac{1+\cos X}{2}}$ in the interval $0 \le X \le 2\pi$. The graphs did *not* coincide, as shown on the right. Explain to Kevin what he did wrong.



CHAPTER SUMMARY

A trigonometric identity can be proved by using the basic identities to change one side of the identity into the form of the other side.

The function values of $(A \pm B)$, $2A$, and $\frac{1}{2}A$, can be written in terms of the function values of A and of B, using the following identities:

Sums of Angle Measures
$\cos(A + B) = \cos A\cos B - \sin A\sin B$
$\sin(A + B) = \sin A\cos B + \cos A\sin B$
$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

Differences of Angle Measures
$\cos(A - B) = \cos A\cos B + \sin A\sin B$
$\sin(A - B) = \sin A\cos B - \cos A\sin B$
$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

Double-Angle Formulas
$\sin(2A) = 2\sin A\cos A$
$\cos(2A) = \cos^2 A - \sin^2 A$
$\cos(2A) = 2\cos^2 A - 1$
$\cos(2A) = 1 - 2\sin^2 A$
$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$

Half-Angle Formulas
$\sin\frac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{2}}$
$\cos\frac{1}{2}A = \pm\sqrt{\frac{1 + \cos A}{2}}$
$\tan\frac{1}{2}A = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$

VOCABULARY

12-7 Double-angle formulas

12-8 Half-angle formulas

REVIEW EXERCISES

In 1–6, prove each identity.

1. $\sec\theta = \csc\theta\tan\theta$

2. $\cos\theta\cot\theta + \sin\theta = \csc\theta$

3. $2\sin^2\theta = 1 - \cos 2\theta$

4. $\tan\theta + \frac{1}{\csc\theta} = \frac{1 + \cos\theta}{\cot\theta}$

5. $\frac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} = \tan\theta$

6. $\frac{\sin(A + B) + \sin(A - B)}{\sin(A + B) - \sin(A - B)} = \tan A\cot B$

In 7–21, $\sin A = -\frac{7}{25}$, $\sin B = -\frac{3}{5}$, and both A and B are third-quadrant angles. Find each function value.

7. $\cos A$ **8.** $\cos B$ **9.** $\tan A$

10. $\tan B$ **11.** $\sin(A + B)$ **12.** $\sin(A - B)$

13. $\cos(A + B)$ **14.** $\cos(A - B)$ **15.** $\tan(A + B)$

16. $\tan(A - B)$ **17.** $\sin 2A$ **18.** $\cos 2B$

19. $\tan 2A$ **20.** $\sin\frac{1}{2}A$ **21.** $\cos\frac{1}{2}A$

22. If $\cos A = 0.2$ and A and B are complementary angles, find $\cos B$.

23. If $\sin A = 0.6$ and A and B are supplementary angles, find $\cos B$.

In 24–32, θ is the measure of an acute angle and $\sin\theta = \frac{\sqrt{7}}{4}$. Find each function value.

24. $\cos\theta$ **25.** $\tan\theta$ **26.** $\sin 2\theta$

27. $\cos 2\theta$ **28.** $\tan 2\theta$ **29.** $\sin\frac{1}{2}\theta$

30. $\cos\frac{1}{2}\theta$ **31.** $\tan\frac{1}{2}\theta$ **32.** $\cos(2\theta + \theta)$

In 33–41, $360° < \theta < 450°$ and $\tan\theta = \frac{1}{7}$. Find, in simplest radical form, each function value.

33. $\sec\theta$ **34.** $\cos\theta$ **35.** $\sin\theta$

36. $\tan\theta$ **37.** $\sin 2\theta$ **38.** $\cos 2\theta$

39. $\tan 2\theta$ **40.** $\sin\frac{1}{2}\theta$ **41.** $\cos\frac{1}{2}\theta$

42. Show that if A and B are complementary, $\cos A\cos B = \sin A\sin B$.

Exploration

In this activity, you will derive the **triple-angle formulas**.

$$\sin(3A) = 3\sin A - 4\sin^3 A$$

$$\cos(3A) = 4\cos^3 A - 3\cos A$$

$$\tan(3A) = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

1. Examine the graphs of these equations on the graphing calculator in the interval $0 \leq A \leq 2\pi$. Explain why these equations appear to be identities.

2. Use $2A + A = 3A$ to prove that each equation is an identity.

CUMULATIVE REVIEW CHAPTERS 1–12

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The solution set of $2 - \sqrt{x + 3} = 6$ is

(1) $\{1\}$ (2) $\{-19\}$ (3) $\{13\}$ (4) $\varnothing$

2. The radian measure of an angle of 240° is

(1) $\frac{4\pi}{3}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{6}$ (4) $\frac{7\pi}{6}$

3. If $3^x = 27^{\frac{2}{3}}$, then x is equal to

(1) 1 (2) 2 (3) 3 (4) 4

4. When expressed in $a + bi$ form, $\left(12 + \sqrt{-9}\right) - \left(3 - \sqrt{4}\right)$ is equal to

(1) $10 + 3i$ (2) $9 + 5i$ (3) $11 + 3i$ (4) $9 + i$

5. $\sum_{n=0}^{3} 2^n$ is equal to

(1) 8 (2) 9 (3) 14 (4) 15

6. The fraction $\frac{3 + \sqrt{7}}{3 - \sqrt{7}}$ is equal to

(1) $8 + 3\sqrt{7}$ (3) $8 + \sqrt{7}$
(2) 8 (4) $8 + \frac{1}{2}\sqrt{7}$

7. If $f(x) = (x + 2)^2$ and $g(x) = x - 1$, then $g(f(x)) =$

(1) $(x + 2)^2 - 1$ (3) $x^2 + 5x + 3$
(2) $(x + 1)^2$ (4) $(2x + 1)^2$

8. The solution set of $2x^2 - 5x = 3$ is

(1) $\left\{\frac{1}{2}, -3\right\}$ (3) $\left\{\frac{3}{2}, 1\right\}$
(2) $\left\{-\frac{1}{2}, 3\right\}$ (4) $\left\{-\frac{3}{2}, -1\right\}$

9. When $\log x = 2 \log A - \frac{1}{2} \log B$, x is equal to

(1) $2A - \frac{1}{2}B$ (3) $\frac{A^2}{\sqrt{B}}$
(2) $A^2 - \sqrt{B}$ (4) $\frac{2A}{\frac{1}{2}B}$

10. If $y = \arccos 0$, then y is equal to

(1) 0 (2) 1 (3) $\frac{\pi}{2}$ (4) π

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Write the multiplicative inverse of $1 + i$ in $a + bi$ form.

12. For what values of c and a does $x^2 + 5x + c = (x + a)^2$? Justify your answer.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Solve $x^2 + 3x - 10 \geq 0$ for x and graph the solution set on a number line.

14. Write the equation of a circle if the endpoints of a diameter of the circle are (0, –1) and (2, 5).

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. **a.** Sketch the graph of $f(x) = \sin x$ in the interval $-2\pi \leq x \leq 2\pi$.

b. On the same set of axes, sketch the graph of $g(x) = f\left(x + \frac{\pi}{4}\right) + 3$ and write an equation for $g(x)$.

16. The first term of a geometric sequence is 1 and the fifth term is 9.

a. What is the common ratio of the sequence?

b. Write the first eight terms of the sequence.

c. Write the sum of the first eight terms of the sequence using sigma notation.