CHAPTER

10

MORE TRIGONOMETRIC FUNCTIONS

The relationships among the lengths of the sides of an isosceles right triangle or of the right triangles formed by the altitude to a side of an equilateral triangle make it possible for us to find exact values of the trig functions of angles of 30°, 45°, and their multiples. How are the trigonometric function values for other angles determined? Before calculators and computers were readily available, handbooks that contained tables of values were a common tool of mathematicians. Now calculators or computers will return these values. How were the values in these tables or those displayed by a calculator determined?

In more advanced math courses, you will learn that trigonometric function values can be approximated by evaluating an infinite series to the required number of decimal places.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

To use this formula correctly, x must be expressed in a unit of measure that has no dimension. This unit of measure, which we will define in this chapter, is called a *radian*.

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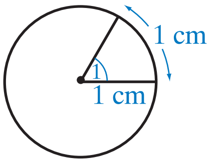
10-1 RADIAN MEASURE

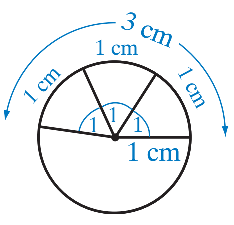
A line segment can be measured in different units of measure such as feet, inches, centimeters, and meters. Angles and arcs can also be measured in different units of measure. We have measured angles in degrees, but another frequently used unit of measure for angles is called a *radian*.

DEFINITION

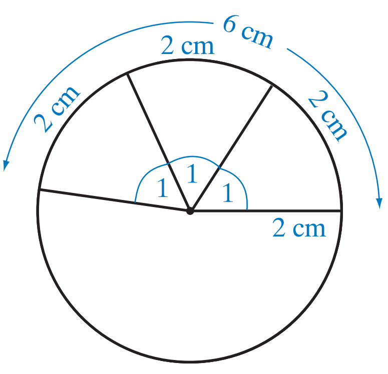
A **radian** is the unit of measure of a central angle that intercepts an arc equal in length to the radius of the circle.



In a circle of radius 1 centimeter, a central angle whose measure is 1 radian intercepts an arc whose length is 1 centimeter.



The radian measure of any central angle in a circle of radius 1 is equal to the length of the arc that it intercepts. In a circle of radius 1 centimeter, a central angle of 2 radians intercepts an arc whose length is 2 centimeters, and a central angle of 3 radians intercepts an arc whose length is 3 centimeters.

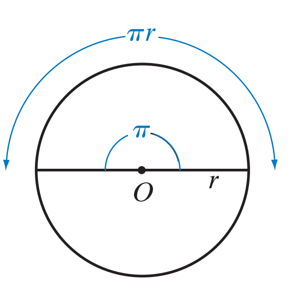


In a circle of radius 2 centimeters, a central angle whose measure is 1 radian intercepts an arc whose length is 2 centimeters, a central angle of 2 radians intercepts an arc whose measure is 4 centimeters, and a central angle of 3 radians intercepts an arc whose measure is 6 centimeters.

In general, the radian measure θ of a central angle of a circle is the length of the intercepted arc, s, divided by the radius of the circle, r.

$$\theta = \frac{s}{r}$$

Relationship Between Degrees and Radians



When the diameter of a circle is drawn, the straight angle whose vertex is at the center of the circle intercepts an arc that is a semicircle. The length of the semicircle is one-half the circumference of the circle or $\frac{1}{2}(2\pi r) = \pi r$. Therefore, the radian measure of a straight angle is $\frac{\pi r}{r} = \pi$. Since the degree measure of a straight angle is 180°, π radians and 180 degrees are measures of the same angle. We can write the following relationship:

$$\boldsymbol{\pi} \textbf{ radians} = \textbf{180 degrees}$$

We will use this relationship to change the degree measure of any angle to radian measure.

Changing Degrees to Radians

The measure of any angle can be expressed in degrees or in radians. We can do this by forming the following proportion:

$$\frac{\textbf{measure in degrees}}{\textbf{measure in radians}} = \frac{\mathbf{180}}{\boldsymbol{\pi}}$$

This proportion can be used to find the radian measure of an angle whose degree measure is known. For example, to find θ, the radian measure of an angle of 30°, substitute 30 for "measure in degrees" and θ for "measure in radians" in the proportion. Then use the fact that the product of the means is equal to the product of the extremes to solve for θ.

$$\begin{aligned} \frac{30}{\theta} &= \frac{180}{\pi} \\ 180\theta &= 30\pi \\ \theta &= \frac{30\pi}{180} \\ \theta &= \frac{\pi}{6} \end{aligned}$$

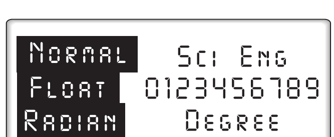
The radian measure of an angle of 30° is $\frac{\pi}{6}$.

We know that $\sin 30° = \frac{1}{2}$. Use a calculator to compare this value to $\sin \theta$ when θ is $\frac{\pi}{6}$ radians.

Begin by changing the calculator to radian mode.

ENTER: MODE ▼ ▼ ENTER

DISPLAY:



Then enter the expression.

ENTER: SIN 2nd π ÷ 6) MATH ENTER ENTER

DISPLAY:

The calculator confirms that $\sin 30° = \sin \frac{\pi}{6}$.

Changing Radians to Degrees

When the radian measure of an angle is known, we can use either of the two methods, shown below, to find the degree measure of the angle.

METHOD 1 *Proportion*

Use the same proportion that was used to change from degrees to radians. For example, to convert an angle of $\frac{\pi}{2}$ radians to degrees, let θ = degree measure of the angle.

(1) Write the proportion that relates degree measure to radian measure:
$$\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{180}{\pi}$$

(2) Substitute θ for the degree measure and $\frac{\pi}{2}$ for the radian measure:
$$\frac{\theta}{\left(\frac{\pi}{2}\right)} = \frac{180}{\pi}$$

(3) Solve for θ:
$$\pi\theta = \frac{\pi}{2}(180)$$
$$\pi\theta = 90\pi$$
$$\theta = 90$$

The measure of the angle is 90°.

METHOD 2 *Substitution*

Since we know that π radians = 180°, substitute 180° for π radians and simplify the expression. For example:

$$\frac{\pi}{2}\text{ radians} = \frac{1}{2}(\pi\text{ radians}) = \frac{1}{2}(180°) = 90°$$

Radian measure is a ratio of the length of an arc to the length of the radius of a circle. These ratios must be expressed in the same unit of measure. Therefore, the units cancel and the ratio is a number that has no unit of measure.

EXAMPLE 1

The radius of a circle is 4 centimeters. A central angle of the circle intercepts an arc of 12 centimeters.

a. What is the radian measure of the angle?

b. Is the angle acute, obtuse, or larger than a straight angle?

Solution **a.** *How to Proceed*

(1) Write the rule that states that the radian measure of a central angle, θ, is equal to the length s of the intercepted arc divided by the length r of a radius: $\theta = \frac{s}{r}$

(2) Substitute the given values: $\theta = \frac{12}{4}$

(3) Simplify the fraction: $\theta = 3$

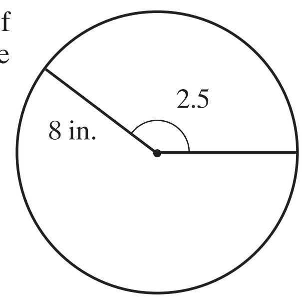
b. Since we know that π radians $= 180°$, then 1 radian $= \frac{180°}{\pi}$.

3 radians $= 3\left(\frac{180°}{\pi}\right) \approx 171.88°$. Therefore, the angle is obtuse.

Answers **a.** 3 radians **b.** Obtuse

EXAMPLE 2

The radius of a circle is 8 inches. What is the length of the arc intercepted by a central angle of the circle if the measure of the angle is 2.5 radians?



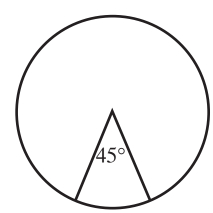
Solution

$$\theta = \frac{s}{r}$$
$$2.5 = \frac{s}{8}$$
$$s = 2.5(8) = 20$$

Answer 20 inches

EXAMPLE 3

What is the radian measure of an angle of 45°?



Solution *How to Proceed*

(1) Write the proportion: $\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{180}{\pi}$

(2) Substitute x for the radian measure and 45 for the degree measure: $\frac{45}{x} = \frac{180}{\pi}$

(3) Solve for x: $180x = 45\pi$

$x = \frac{45}{180}\pi$

(4) Express x in simplest form: $x = \frac{1}{4}\pi$

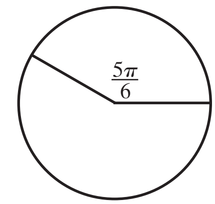
Answer $\frac{1}{4}\pi$ or $\frac{\pi}{4}$

EXAMPLE 4

The radian measure of an angle is $\frac{5\pi}{6}$.

a. What is the measure of the angle in degrees?

b. What is the radian measure of the reference angle?



Solution **a. METHOD 1:** *Proportions*

Let θ = degree measure of the angle.

$$\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{180}{\pi}$$

$$\frac{\theta}{\frac{5\pi}{6}} = \frac{180}{\pi}$$

$$\pi\theta = 180\left(\frac{5\pi}{6}\right)$$

$$\pi\theta = 150\pi$$

$$\theta = 150$$

METHOD 2: *Substitution*

Substitute 180° for π radians, and simplify.

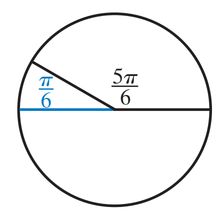
$$\frac{5\pi}{6}\text{ radians} = \frac{5}{6}(\pi\text{ radians})$$

$$= \frac{5}{6}(180^\circ)$$

$$= 150^\circ$$

b. Since $\frac{5\pi}{6}$ is a second-quadrant angle, its reference angle is equal to $180^\circ - 150^\circ$ or, in radians:

$$180^\circ - 150^\circ \rightarrow \pi - \frac{5\pi}{6} = \frac{\pi}{6}$$



Answers **a.** 150° **b.** $\frac{\pi}{6}$

Exercises

Writing About Mathematics

1. Ryan and Rebecca each found the radian measure of a central angle by measuring the radius of the circle and the length of the intercepted arc. Ryan used inches and Rebecca used centimeters when making their measurements. If Ryan and Rebecca each measured accurately, will the measures that they obtain for the angle be equal? Justify your answer.

2. If a wheel makes two complete revolutions, each spoke on the wheel turns through an angle of how many radians? Explain your answer.

Developing Skills

In 3–12, find the radian measure of each angle whose degree measure is given.

3. 30° **4.** 90° **5.** 45° **6.** 120° **7.** 160°

8. 135° **9.** 225° **10.** 240° **11.** 270° **12.** 330°

In 13–22, find the degree measure of each angle whose radian measure is given.

13. $\frac{\pi}{3}$ **14.** $\frac{\pi}{9}$ **15.** $\frac{\pi}{10}$ **16.** $\frac{2\pi}{5}$ **17.** $\frac{10\pi}{9}$

18. $\frac{3\pi}{2}$ **19.** 3π **20.** $\frac{11\pi}{6}$ **21.** $\frac{7\pi}{2}$ **22.** 1

In 23–27, for each angle with the given radian measure: **a.** Give the measure of the angle in degrees. **b.** Give the measure of the reference angle in radians. **c.** Draw the angle in standard position and its reference angle as an acute angle formed by the terminal side of the angle and the x-axis.

23. $\frac{\pi}{3}$ **24.** $\frac{7\pi}{36}$ **25.** $\frac{10\pi}{9}$ **26.** $-\frac{7\pi}{18}$ **27.** $\frac{25\pi}{9}$

In 28–37, θ is the radian measure of a central angle that intercepts an arc of length s in a circle with a radius of length r.

28. If $s = 6$ and $r = 1$, find θ.

29. If $\theta = 4.5$ and $s = 9$, find r.

30. If $\theta = 2.5$ and $r = 10$, find s.

31. If $r = 2$ and $\theta = 1.6$, find s.

32. If $r = 2.5$ and $s = 15$, find θ.

33. If $s = 16$ and $\theta = 0.4$, find r.

34. If $r = 4.2$ and $s = 21$, find θ.

35. If $r = 6$ and $\theta = \frac{2\pi}{3}$, find s.

36. If $s = 18$ and $\theta = \frac{6\pi}{5}$, find r.

37. If $\theta = 6\pi$ and $r = 1$, find s.

38. Circle O has a radius of 1.7 inches. What is the length, in inches, of an arc intercepted by a central angle whose measure is 2 radians?

39. In a circle whose radius measures 5 feet, a central angle intercepts an arc of length 12 feet. Find the radian measure of the central angle.

40. The central angle of circle O has a measure of 4.2 radians and it intercepts an arc whose length is 6.3 meters. What is the length, in meters, of the radius of the circle?

41. Complete the following table, expressing degree measures in radian measure in terms of π.

Degrees	30°	45°	60°	90°	180°	270°	360°
Radians							

Applying Skills

42. The pendulum of a clock makes an angle of 2.5 radians as its tip travels 18 feet. What is the length of the pendulum?

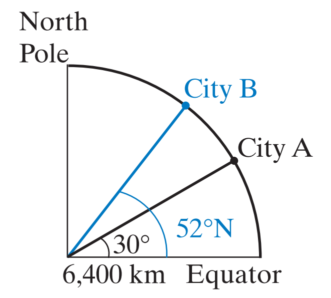
43. A wheel whose radius measures 16 inches is rotated. If a point on the circumference of the wheel moves through an arc of 12 feet, what is the measure, in radians, of the angle through which a spoke of the wheel travels?

44. The wheels on a bicycle have a radius of 40 centimeters. The wheels on a cart have a radius of 10 centimeters. The wheels of the bicycle and the wheels of the cart all make one complete revolution.

a. Do the wheels of the bicycle rotate through the same angle as the wheels of the cart? Justify your answer.

b. Does the bicycle travel the same distance as the cart? Justify your answer.

45. Latitude represents the measure of a central angle with vertex at the center of the earth, its initial side passing through a point on the equator, and its terminal side passing through the given location. (See the figure.) Cities A and B are on a north-south line. City A is located at 30°N and City B is located at 52°N. If the radius of the earth is approximately 6,400 kilometers, find d, the distance between the two cities along the circumference of the earth. Assume that the earth is a perfect sphere.



10-2 TRIGONOMETRIC FUNCTION VALUES AND RADIAN MEASURE

Using Radians to Find Trigonometric Function Values

Hands-On Activity

In previous chapters, we found the exact trigonometric function values of angles of 0°, 30°, 45°, 60°, 90°, 180°, and 270°. Copy and complete the table below to show measures of these angles in radians and the corresponding function values. Let θ be the measure of an angle in standard position.

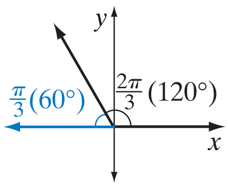
θ in Degrees	0°	30°	45°	60°	90°	180°	270°
θ in Radians							
sin θ							
cos θ							
tan θ							

We can use these function values to find the exact function value of any angle whose radian measure is a multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

EXAMPLE 1

Find the exact value of tan $\frac{2\pi}{3}$.

Solution



METHOD 1

The radian measure $\frac{2\pi}{3}$ is equivalent to the degree measure $\frac{2(180°)}{3} = 120°$.

An angle of 120° is a second-quadrant angle whose tangent is negative.

The reference angle is $180° - 120° = 60°$.

$$\tan \frac{2\pi}{3} = \tan 120° = -\tan 60° = -\sqrt{3}$$

METHOD 2

Since $\frac{\pi}{2} < \frac{2\pi}{3} < \pi$, an angle of $\frac{2\pi}{3}$ is a second-quadrant angle.

The reference angle of an angle of $\frac{2\pi}{3}$ radians is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

The tangent of a second-quadrant angle is negative.

$$\tan \frac{2\pi}{3} = -\tan \frac{\pi}{3} = -\sqrt{3}$$

Answer $-\sqrt{3}$

The graphing calculator will return the trigonometric function values of angles entered in degree or radian measure. Before entering an angle measure, be sure that the calculator is in the correct mode. For example, to find $\tan \frac{\pi}{8}$ to four decimal places, use the following sequence of keys and round the number given by the calculator. Be sure that your calculator is set to radian mode.

ENTER: TAN 2nd π ÷ 8) ENTER

DISPLAY:
```
TAN(π/8)
        .4142135624
```

Therefore, $\tan \frac{\pi}{8} \approx 0.4142$.

EXAMPLE 2

Find the value of $\cos \theta$ to four decimal places when $\theta = 2.75$ radians.

Solution ENTER: COS 2.75) ENTER

DISPLAY:
```
cos(2.75)
       -.9243023786
```

Note that the cosine is negative because the angle is a second-quadrant angle.

Answer -0.9243

Finding Angle Measures in Radians

When the calculator is in radian mode, it will return the decimal value of the radian measure whose function value is entered. The measure will *not* be in terms of π.

For instance, we know that $\tan 45° = 1$. Since $45 = \frac{180}{4}$, an angle of 45° is an angle of $\frac{\pi}{4}$ radians. Use a calculator to find θ in radians when $\tan \theta = 1$. Your calculator should be in radian mode.

ENTER: 2nd TAN⁻¹ 1) ENTER DISPLAY: TAN⁻¹(1)
.7853981634

The value 0.7853981634 returned by the calculator is approximately equal to $\frac{\pi}{4}$. We can verify this by dividing π by 4 and comparing the answers.

The exact radian measure of the angle is the irrational number $\frac{\pi}{4}$. The rational approximation of the radian measure of the angle is 0.7853981634.

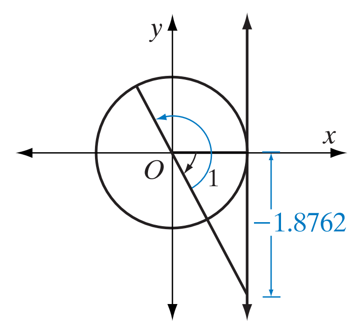
EXAMPLE 3

Find, in radians, the smallest positive value of θ if $\tan \theta = -1.8762$. Express the answer to the nearest hundredth.

Solution The calculator should be in radian mode.

ENTER: 2nd TAN⁻¹ -1.8762) ENTER DISPLAY: TAN⁻¹(⁻1.8762)
⁻1.081104612

Tangent is negative in the second and fourth quadrants. The calculator has returned the measure of the negative rotation in the fourth quadrant. The angle in the second quadrant with a positive measure is formed by the opposite ray and can be found by adding π, the measure of a straight angle, to the given measure.



ENTER: 2nd ANS + 2nd π ENTER DISPLAY: Ans+π
2.060488041

Check Verify this answer by finding tan 2.060488041.

ENTER: TAN 2nd ANS) ENTER DISPLAY: TAN(Ans)
⁻1.8762

Answer 2.06

EXAMPLE 4

Find, to the nearest ten-thousandth, a value of θ in radians if $\csc\theta = 2.369$.

Solution If $\csc\theta = 2.369$, then $\sin\theta = \frac{1}{2.369}$.

ENTER: 2nd SIN⁻¹ 1 ÷ 2.369) ENTER

DISPLAY: sin⁻¹(1/2.369)
.4357815508

Rounded to four decimal places, $\theta = 0.4358$. *Answer*

Exercises

Writing About Mathematics

1. Alexia said that when θ is a second-quadrant angle whose measure is in radians, the measure of the reference angle in radians is $\pi - \theta$. Do you agree with Alexia? Explain why or why not.

2. Diego said that when θ is the radian measure of an angle, the angle whose radian measure is $2\pi n + \theta$ is an angle with the same terminal side for all integral values of n. Do you agree with Diego? Explain why or why not.

Developing Skills

In 3–12, find the exact function value of each of the following if the measure of the angle is given in radians.

3. $\sin\frac{\pi}{4}$ **4.** $\tan\frac{\pi}{3}$ **5.** $\cos\frac{\pi}{2}$ **6.** $\tan\frac{\pi}{6}$ **7.** $\cos\frac{2\pi}{3}$

8. $\sin\frac{4\pi}{3}$ **9.** $\tan\frac{5\pi}{4}$ **10.** $\sec\frac{\pi}{3}$ **11.** $\csc\pi$ **12.** $\cot\frac{\pi}{4}$

In 13–24, find, to the nearest ten-thousandth, the radian measure θ of a first-quadrant angle with the given function value.

13. $\sin\theta = 0.2736$ **14.** $\cos\theta = 0.5379$ **15.** $\tan\theta = 3.726$

16. $\cos\theta = 0.9389$ **17.** $\sin\theta = 0.8267$ **18.** $\cos\theta = 0.8267$

19. $\tan\theta = 1.5277$ **20.** $\cot\theta = 1.5277$ **21.** $\sec\theta = 5.232$

22. $\cot\theta = 0.3276$ **23.** $\csc\theta = 2.346$ **24.** $\cot\theta = 0.1983$

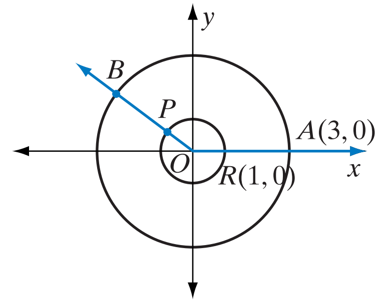
25. If $f(x) = \sin\left(\frac{1}{3}x\right)$, find $f\left(\frac{\pi}{2}\right)$. **26.** If $f(x) = \cos 2x$, find $f\left(\frac{3\pi}{4}\right)$.

27. If $f(x) = \sin 2x + \cos 3x$, find $f\left(\frac{\pi}{4}\right)$. **28.** If $f(x) = \tan 5x - \sin 2x$, find $f\left(\frac{\pi}{6}\right)$.

Applying Skills

29. The unit circle intersects the x-axis at $R(1, 0)$ and the terminal side of $\angle ROP$ at P. What are the coordinates of P if $m\overset{\frown}{RP} = 4.275$?

30. The x-axis intersects the unit circle at $R(1, 0)$ and a circle of radius 3 centered at the origin at $A(3, 0)$. The terminal side of $\angle ROP$ intersects the unit circle at P and the circle of radius 3 at B. The measure of $\overset{\frown}{RP}$ is 2.50.



a. What is the radian measure of $\angle ROP$?

b. What is the radian measure of $\angle AOB$?

c. What are the coordinates of P?

d. What are the coordinates of B?

e. In what quadrant do P and B lie? Justify your answer.

31. The wheels of a cart that have a radius of 12 centimeters move in a counterclockwise direction for 20 meters.

a. What is the radian measure of the angle through which each wheel has turned?

b. What is sine of the angle through which the wheels have turned?

32. A supporting cable runs from the ground to the top of a tree that is in danger of falling down. The tree is 18 feet tall and the cable makes an angle of $\frac{2\pi}{9}$ with the ground. Determine the length of the cable to the nearest tenth of a foot.

33. An airplane climbs at an angle of $\frac{\pi}{15}$ with the ground. When the airplane has reached an altitude of 500 feet:

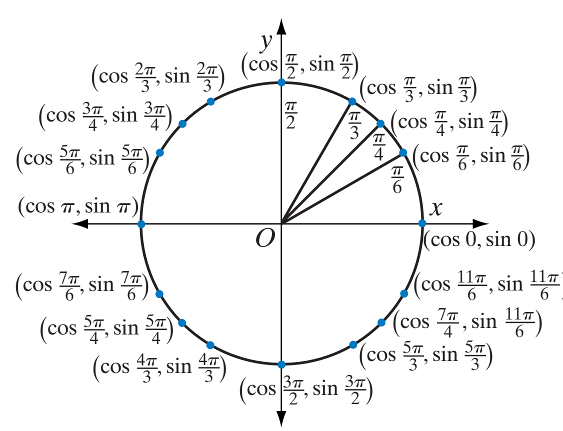
a. What is the distance in the air that the airplane has traveled?

b. What is the horizontal distance that the airplane has traveled?

Hands-On Activity 1: The Unit Circle and Radian Measure

In Chapter 9, we explored trigonometric function values on the unit circle with regard to degree measure. In the following Hands-On Activity, we will examine trigonometric function values on the unit circle with regard to *radian measure*.

1. Recall that for any point P on the unit circle, $\overrightarrow{OP}$ is the terminal side of an angle in standard position. If the measure of this angle is θ, the coordinates of P are $(\cos\theta, \sin\theta)$. Use your knowledge of the *exact* cosine and sine values for $0°$, $30°$, $45°$, $60°$, and $90°$ angles to find the coordinates of the first-quadrant points.



2. The cosine and sine values of the other angles marked on the unit circle can be found by relating them to the points in the first quadrant or along the positive axes. For example, the point $\left(\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}\right)$ is the image of the point $\left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right)$ under a reflection in the y-axis. Thus, if $\left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right) = (a, b)$, then

$$\left(\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}\right) = (-a, b).$$

Determine the *exact* coordinates of the points shown on the unit circle to find the function values.

Hands-On Activity 2: Evaluating the Sine and Cosine Functions

The functions sin x and cos x can be represented by the following series when x is in radians:

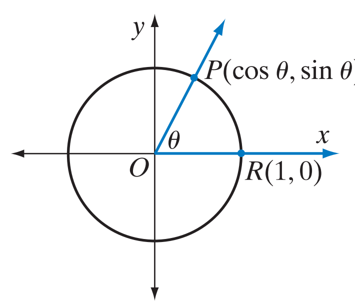
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

1. Write the next two terms for each series given above.

2. Use the first six terms of the series for sin x to approximate $\sin \frac{\pi}{4}$ to four decimal places.

3. Use the first six terms of the series for cos x to approximate $\cos \frac{\pi}{3}$ to four decimal places.

10-3 PYTHAGOREAN IDENTITIES



The unit circle is a circle of radius 1 with center at the origin. Therefore, the equation of the unit circle is $x^2 + y^2 = 1^2$ or $x^2 + y^2 = 1$. Let P be any point on the unit circle and $\overrightarrow{OP}$ be the terminal side of $\angle ROP$, an angle in standard position whose measure is θ. The coordinates of P are $(\cos \theta, \sin \theta)$. Since P is a point on the circle whose equation is $x^2 + y^2 = 1$, and $x = \cos \theta$ and $y = \sin \theta$:

$$(\cos \theta)^2 + (\sin \theta)^2 = 1$$

In order to emphasize that it is the cosine value and the sine value that are being squared, we write $(\cos \theta)^2$ as $\cos^2 \theta$ and $(\sin \theta)^2$ as $\sin^2 \theta$. Therefore, the equation is written as:

$$\cos^2 \theta + \sin^2 \theta = 1$$

This equation is called an *identity*.

DEFINITION

An **identity** is an equation that is true for all values of the variable for which the terms of the variable are defined.

Because the identity $\cos^2 \theta + \sin^2 \theta = 1$ is based on the Pythagorean theorem, we refer to it as a **Pythagorean identity**.

EXAMPLE 1

Verify that $\cos^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{3} = 1$.

Solution We know that $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Therefore:

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{1}{4} + \frac{3}{4} &= 1 \\ 1 &= 1 \checkmark \end{aligned}$$

We can write two related Pythagorean identities by dividing both sides of the equation by the same expression.

Divide by $\cos^2 \theta$:

Recall that $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\sec \theta = \frac{1}{\cos \theta}$.

For all values of θ for which $\cos \theta \neq 0$:

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} &= \frac{1}{\cos^2 \theta} \\ 1 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 &= \left(\frac{1}{\cos \theta}\right)^2 \\ 1 + \tan^2 \theta &= \sec^2 \theta \end{aligned}$$

$$\mathbf{1 + \tan^2 \theta = \sec^2 \theta}$$

Divide by $\sin^2 \theta$:

Recall that $\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$ and $\csc \theta = \frac{1}{\sin \theta}$.

For all values of θ for which $\sin \theta \neq 0$:

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} &= \frac{1}{\sin^2 \theta} \\ \left(\frac{\cos \theta}{\sin \theta}\right)^2 + 1 &= \left(\frac{1}{\sin \theta}\right)^2 \end{aligned}$$

$$\mathbf{\cot^2 \theta + 1 = \csc^2 \theta}$$

EXAMPLE 2

If $\cos \theta = \frac{1}{3}$ and θ is in the fourth quadrant, use an identity to find:

a. $\sin \theta$ **b.** $\tan \theta$ **c.** $\sec \theta$ **d.** $\csc \theta$ **e.** $\cot \theta$

Solution Since θ is in the fourth quadrant, cosine and its reciprocal, secant, are positive; sine and tangent and their reciprocals, cosecant and cotangent, are negative.

a. $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{1}{3}\right)^2 + \sin^2\theta = 1$$

$$\frac{1}{9} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{8}{9}$$

$$\sin\theta = -\sqrt{\frac{8}{9}}$$

$$\sin\theta = -\frac{2\sqrt{2}}{3}$$

b. $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = -2\sqrt{2}$

c. $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{3}} = 3$

d. $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{2\sqrt{2}}{3}} = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$

e. $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-2\sqrt{2}} = -\frac{\sqrt{2}}{4}$

Answers **a.** $-\frac{2\sqrt{2}}{3}$ **b.** $-2\sqrt{2}$ **c.** 3 **d.** $-\frac{3\sqrt{2}}{4}$ **e.** $-\frac{\sqrt{2}}{4}$

EXAMPLE 3

Write $\tan\theta + \cot\theta$ in terms of $\sin\theta$ and $\cos\theta$ and write the sum as a single fraction in simplest form.

Solution

$$\begin{aligned}
\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&= \frac{\sin\theta}{\cos\theta} \times \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\cos\theta} \\
&= \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \\
&= \frac{1}{\cos\theta\sin\theta} \quad \textit{Answer}
\end{aligned}$$

SUMMARY

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cos^2\theta + \sin^2\theta = 1$
$\csc\theta = \frac{1}{\sin\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$1 + \tan^2\theta = \sec^2\theta$
$\cot\theta = \frac{1}{\tan\theta}$		$\cot^2\theta + 1 = \csc^2\theta$

Exercises

Writing About Mathematics

1. Emma said that the answer to Example 3 can be simplified to $(\sec\theta)(\csc\theta)$. Do you agree with Emma? Justify your answer.

2. Ethan said that the equations $\cos^2\theta = 1 - \sin^2\theta$ and $\sin^2\theta = 1 - \cos^2\theta$ are identities. Do you agree with Ethan? Explain why or why not.

Developing Skills

In 3–14, for each given function value, find the remaining five trigonometric function values.

3. $\sin\theta = \frac{1}{5}$ and θ is in the second quadrant.

4. $\cos\theta = \frac{3}{4}$ and θ is in the first quadrant.

5. $\cos\theta = -\frac{3}{4}$ and θ is in the third quadrant.

6. $\sin\theta = -\frac{2}{3}$ and θ is in the fourth quadrant.

7. $\sin\theta = \frac{2}{3}$ and θ is in the second quadrant.

8. $\tan\theta = -2$ and θ is in the second quadrant.

9. $\tan\theta = 4$ and θ is in the third quadrant.

10. $\sec\theta = -8$ and θ is in the second quadrant.

11. $\cot\theta = \frac{5}{3}$ and θ is in the third quadrant.

12. $\csc\theta = \frac{5}{4}$ and θ is in the second quadrant.

13. $\sin\theta = \frac{4}{5}$ and θ is in the second quadrant.

14. $\cot\theta = -6$ and θ is in the fourth quadrant.

In 15–22, write each given expression in terms of sine and cosine and express the result in simplest form.

15. $(\csc\theta)(\sin\theta)$

16. $(\sin\theta)(\cot\theta)$

17. $(\tan\theta)(\cos\theta)$

18. $(\sec\theta)(\cot\theta)$

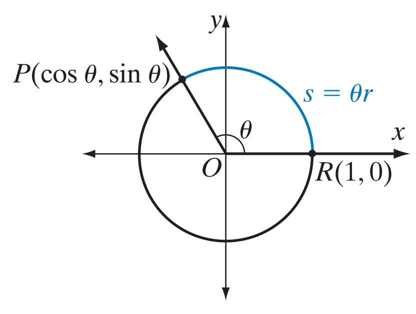
19. $\sec\theta + \tan\theta$

20. $\frac{\sec\theta}{\csc\theta}$

21. $\csc^2\theta - \frac{\cot\theta}{\tan\theta}$

22. $\sec\theta\,(1 + \cot\theta) - \csc\theta\,(1 + \tan\theta)$

10-4 DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS



In Chapter 9, we defined the trigonometric functions for degree measures on the unit circle. Using arc length, we can define the trigonometric functions on the set of real numbers. Recall that the function values of sine and cosine are given by the coordinates of P, that is, $P(\cos\theta, \sin\theta)$. The length of the arc intercepted by the angle is given by

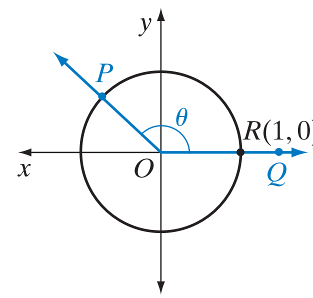
$$\theta = \frac{s}{r} \qquad \text{or} \qquad s = \theta r$$

However, since the radius of the unit circle is 1, the length of the arc s is equal to θ. In other words, the length of the arc s is equivalent to the measure of the angle θ in radians, a real number. Thus, any trigonometric function of θ is a function on the set of real numbers.

The Sine and Cosine Functions

We have defined the sine function and the cosine function in terms of the measure of angle formed by a rotation. The measure is positive if the rotation is in the counterclockwise direction and negative if the rotation is in the clockwise rotation. The rotation can continue indefinitely in both directions.

► **The domain of the sine function and of the cosine function is the set of real numbers.**



To find the range, let P be any point on the unit circle and $\overrightarrow{OP}$ be the terminal side of $\angle ROP$, an angle in standard position with measure θ. The coordinates of P are $(\cos\theta, \sin\theta)$. Since the unit circle has a radius of 1, as P moves on the circle, its coordinates are elements of the interval $[-1, 1]$. Therefore, the range of $\cos\theta$ and $\sin\theta$ is the set of real numbers $[-1, 1]$.

To see this algebraically, consider the identity $\cos^2\theta + \sin^2\theta = 1$. Subtract $\cos^2\theta$ from both sides of this identity and solve for $\sin\theta$. Similarly, subtract $\sin^2\theta$ from both sides of this identity and solve for $\cos\theta$.

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \sin^2\theta &= 1 - \cos^2\theta \\ \sin\theta &= \pm\sqrt{1 - \cos^2\theta} \end{aligned} \qquad \begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \cos^2\theta &= 1 - \sin^2\theta \\ \cos\theta &= \pm\sqrt{1 - \sin^2\theta} \end{aligned}$$

The value of $\sin\theta$ will be a real number if and only if $1 - \cos^2\theta \geq 0$, that is, if and only if $|\cos\theta| \leq 1$ or $-1 \leq \cos\theta \leq 1$.

The value of $\cos\theta$ will be a real number if and only if $1 - \sin^2\theta \geq 0$, that is, if and only if $|\sin\theta| \leq 1$ or $-1 \leq \sin\theta \leq 1$.

► **The range of the sine function and of the cosine function is the set of real numbers $[-1, 1]$, that is, the set of real numbers from -1 to 1, including -1 and 1.**

The Secant Function

The secant function is defined for all real numbers such that $\cos\theta \neq 0$. We know that $\cos\theta = 0$ when $\theta = \frac{\pi}{2}$, when $\theta = \frac{3\pi}{2}$, and when θ differs from either of these values by a complete rotation, 2π.

Therefore, $\cos\theta = 0$ when θ equals:

$\frac{\pi}{2}$	$\frac{3\pi}{2} = \frac{\pi}{2} + \pi$
$\frac{\pi}{2} + 2\pi$	$\frac{3\pi}{2} + 2\pi = \frac{\pi}{2} + 3\pi$
$\frac{\pi}{2} + 4\pi$	$\frac{3\pi}{2} + 4\pi = \frac{\pi}{2} + 5\pi$
$\frac{\pi}{2} + 6\pi$	$\frac{3\pi}{2} + 6\pi = \frac{\pi}{2} + 7\pi$
$\vdots$	$\vdots$

We can continue with this pattern, which can be summarized as $\frac{\pi}{2} + n\pi$ for all integral values of n. The domain of the secant function is the set of all real numbers except those for which $\cos\theta = 0$.

► **The domain of the secant function is the set of real numbers except $\frac{\pi}{2} + n\pi$ for all integral values of n.**

The secant function is defined as $\sec\theta = \frac{1}{\cos\theta}$. Therefore:

When $0 < \cos\theta \leq 1$,

$$\frac{0}{\cos\theta} < \frac{\cos\theta}{\cos\theta} \leq \frac{1}{\cos\theta} \qquad \text{or} \qquad 0 < 1 \leq \sec\theta$$

When $-1 \leq \cos\theta < 0$,

$$\frac{-1}{\cos\theta} \geq \frac{\cos\theta}{\cos\theta} > \frac{0}{\cos\theta} \qquad \text{or} \qquad -\sec\theta \geq 1 > 0$$

$$\text{or} \qquad \sec\theta \leq -1 < 0$$

The union of these two inequalities defines the range of the secant function.

► **The range of the secant function is the set of real numbers $(-\infty, -1] \cup [1, \infty)$, that is, the set of real numbers greater than or equal to 1 or less than or equal to -1.**

The Cosecant Function

The cosecant function is defined for all real numbers such that $\sin\theta \neq 0$. We know that $\sin\theta = 0$ when $\theta = 0$, when $\theta = \pi$, and when θ differs from either of these values by a complete rotation, 2π. Therefore, $\sin\theta = 0$ when θ equals:

0	π
$0 + 2\pi = 2\pi$	$\pi + 2\pi = 3\pi$
$0 + 4\pi = 4\pi$	$\pi + 4\pi = 5\pi$
$0 + 6\pi = 6\pi$	$\pi + 6\pi = 7\pi$
$\vdots$	$\vdots$

We can continue with this pattern, which can be summarized as $n\pi$ for all integral values of n. The domain of the cosecant function is the set of all real numbers except those for which $\sin\theta = 0$.

▶ **The domain of the cosecant function is the set of real numbers except $n\pi$ for all integral values of n.**

The range of the cosecant function can be found using the same procedure as was used to find the range of the secant function.

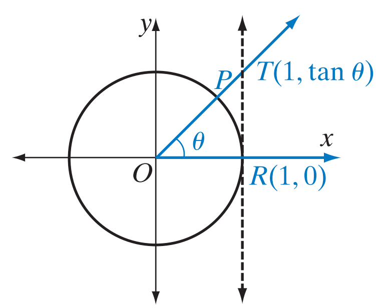
▶ **The range of the cosecant function is the set of real numbers $(-\infty, -1] \cup [1, \infty)$, that is, the set of real numbers greater than or equal to 1 or less than or equal to -1.**

The Tangent Function

We can use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to determine the domain of the tangent function. Since $\sin\theta$ and $\cos\theta$ are defined for all real numbers, $\tan\theta$ is defined for all real numbers for which $\cos\theta \neq 0$, that is, for all real numbers except $\frac{\pi}{2} + n\pi$.

▶ **The domain of the tangent function is the set of real numbers except $\frac{\pi}{2} + n\pi$ for all integral values of n.**

To find the range of the tangent function, recall that the tangent function is defined on the line tangent to the unit circle at $R(1, 0)$. In particular, let T be the point where the terminal side of an angle in standard position intersects the line tangent to the unit circle at $R(1, 0)$. If $m\angle ROT = \theta$, then the coordinates of T are $(1, \tan\theta)$. As the point T moves on the tangent line, its y-coordinate can take on any real-number value. Therefore, the range of $\tan\theta$ is the set of real numbers.



To see this result algebraically, solve the identity $1 + \tan^2\theta = \sec^2\theta$ for $\tan^2\theta$.

$$1 + \tan^2\theta = \sec^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1$$

The range of the secant function is $\sec\theta \geq 1$ or $\sec\theta \leq -1$. Therefore, $\sec^2\theta \geq 1$. If we subtract 1 from each side of this inequality:

$$\sec^2\theta - 1 \geq 0$$
$$\tan^2\theta \geq 0$$
$$\tan\theta \geq 0 \text{ or } \tan\theta \leq 0$$

▶ **The range of the tangent function is the set of all real numbers.**

The Cotangent Function

We can find the domain and range of the cotangent function by a procedure similar to that used for the tangent function.

We can use the identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$ to determine the domain of the cotangent function. Since $\sin\theta$ and $\cos\theta$ are defined for all real numbers, $\cot\theta$ is defined for all real numbers for which $\sin\theta \neq 0$, that is, for all real numbers except $n\pi$.

► **The domain of the cotangent function is the set of real numbers except $n\pi$ for all integral values of n.**

To find the range of the cotangent function, solve the identity $1 + \cot^2\theta = \csc^2\theta$ for $\cot^2\theta$.

$$1 + \cot^2\theta = \csc^2\theta$$
$$\cot^2\theta = \csc^2\theta - 1$$

The range of the cosecant function is $\csc\theta \geq 1$ or $\csc\theta \leq -1$. Therefore, $\csc^2\theta \geq 1$. If we subtract 1 from each side of this inequality:

$$\csc^2\theta - 1 \geq 0$$
$$\cot^2\theta \geq 0$$
$$\cot\theta \geq 0 \text{ or } \cot\theta \leq 0$$

► **The range of the cotangent function is the set of all real numbers.**

EXAMPLE 1

Explain why that is no value of θ such that $\cos\theta = 2$.

Solution The range of the cosine function is $-1 \leq \cos\theta \leq 1$. Therefore, there is no value of θ for which $\cos\theta$ is greater than 1.

SUMMARY

Function	Domain (n is an integer)	Range
Sine	All real numbers	$[-1, 1]$
Cosine	All real numbers	$[-1, 1]$
Tangent	All real numbers except $\frac{\pi}{2} + n\pi$	All real numbers
Cotangent	All real numbers except $n\pi$	All real numbers
Secant	All real numbers except $\frac{\pi}{2} + n\pi$	$(-\infty, -1] \cup [1, \infty)$
Cosecant	All real numbers except $n\pi$	$(-\infty, -1] \cup [1, \infty)$

Exercises

Writing About Mathematics

1. Nathan said that cot θ is undefined for the values of θ for which csc θ is undefined. Do you agree with Nathan? Explain why or why not.

2. Jonathan said that cot θ is undefined for $\frac{\pi}{2}$ because tan θ is undefined for $\frac{\pi}{2}$. Do you agree with Nathan? Explain why or why not.

Developing Skills

In 3–18, for each function value, write the value or tell why it is undefined. Do not use a calculator.

3. $\sin\frac{\pi}{2}$ **4.** $\cos\frac{\pi}{2}$ **5.** $\tan\frac{\pi}{2}$ **6.** $\sec\frac{\pi}{2}$

7. $\csc\frac{\pi}{2}$ **8.** $\cot\frac{\pi}{2}$ **9.** $\tan\pi$ **10.** $\cot\pi$

11. $\sec\frac{3\pi}{2}$ **12.** $\csc\frac{7\pi}{2}$ **13.** $\tan 0$ **14.** $\cot 0$

15. $\tan\left(-\frac{\pi}{2}\right)$ **16.** $\csc\left(-\frac{9\pi}{2}\right)$ **17.** $\sec\left(-\frac{9\pi}{2}\right)$ **18.** $\cot(-8\pi)$

19. List five values of θ for which sec θ is undefined.

20. List five values of θ for which cot θ is undefined.

10-5 INVERSE TRIGONOMETRIC FUNCTIONS

Inverse Sine Function

We know that an angle of 30° is an angle of $\frac{\pi}{6}$ radians and that an angle of 5(30°) or 150° is an angle of $5\left(\frac{\pi}{6}\right)$ or $\frac{5\pi}{6}$. Since $30 = 180 - 150$, an angle of 30° is the reference angle for an angle of 150°. Therefore:

$$\sin 30° = \sin 150° = \frac{1}{2}$$

$$\textit{and} \qquad \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \frac{1}{2}$$

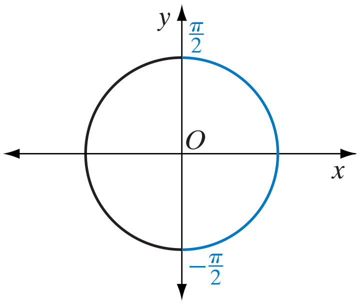
Two ordered pairs of the sine function are $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$. Therefore, the sine function is *not* a one-to-one function. Functions that are not one-to-one do not have an inverse function. When we interchange the elements of the ordered pairs of the sine function, the set of ordered pairs is a relation that is not a function.

$\{(a, b) : b = \sin a\}$ is a function.

$\{(b, a) : a = \arcsin b\}$ is *not* a function.

For every value of θ such that $0 \le \theta \le \frac{\pi}{2}$, there is exactly one nonnegative value of $\sin\theta$. For every value of θ such that $-\frac{\pi}{2} \le \theta < 0$, that is, for every fourth-quadrant angle, there is exactly one negative value of $\sin\theta$.

Therefore, if we **restrict the domain** of the sine function to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, the function is one-to-one and has an inverse function. We designate the **inverse sine function** by *arcsin* or *sin*$^{-1}$.



Sine Function with a Restricted Domain
$y = \sin x$
Domain $= \left\{x : -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$
Range $= \{y : -1 \le y \le 1\}$

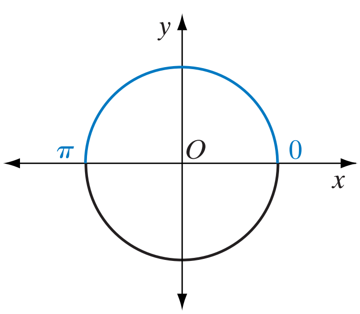
Inverse Sine Function
$y = \arcsin x$ or $y = \sin^{-1} x$
Domain $= \{x : -1 \le x \le 1\}$
Range $= \left\{y : -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}$

Note that when we use the SIN^{-1} key on the graphing calculator with a number in the interval $[-1, 1]$, the response will be an number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Inverse Cosine Function

The cosine function, like the sine function, is *not* a one-to-one function and does not have an inverse function. We can also restrict the domain of the cosine function to form a one-to-one function that has an inverse function.

For every value of θ such that $0 \le \theta \le \frac{\pi}{2}$, there is exactly one nonnegative value of $\cos\theta$. For every value of θ such that $\frac{\pi}{2} < \theta \le \pi$, that is, for every second-quadrant angle, there is exactly one negative value of $\cos\theta$. Therefore, if we restrict the domain of the cosine function to $0 \le \theta \le \pi$, the function is one-to-one and has an inverse function. We designate the **inverse cosine function** by *arccos* or *cos*$^{-1}$.



Cosine Function with a Restricted Domain
$y = \cos x$
Domain $= \{x : 0 \le x \le \pi\}$
Range $= \{y : -1 \le y \le 1\}$

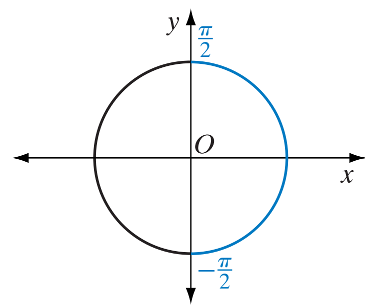
Inverse Cosine Function
$y = \arccos x$ or $y = \cos^{-1} x$
Domain $= \{x : -1 \le x \le 1\}$
Range $= \{y : 0 \le y \le \pi\}$

Note that when we use the COS⁻¹ key on the graphing calculator with a number in the interval $[-1, 1]$, the response will be an number in the interval $[0, \pi]$.

Inverse Tangent Function

The tangent function, like the sine and cosine functions, is *not* a one-to-one function and does not have an inverse function. We can also restrict the domain of the tangent function to form a one-to-one function that has an inverse function.

For every of θ such that $0 \leq \theta < \frac{\pi}{2}$, there is exactly one nonnegative value of $\tan\theta$. For every value of θ such that $-\frac{\pi}{2} < \theta < 0$, that is, for every fourth-quadrant angle, there is exactly one negative value of $\tan\theta$. Therefore, if we restrict the domain of the tangent function to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, the function is one-to-one and has an inverse function. We designate the **inverse tangent function** by *arctan* or *tan⁻¹*.



Tangent Function with a Restricted Domain
$y = \tan x$
Domain $= \left\{x : -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$
Range $= \{y : y \text{ is a real number}\}$

Inverse Tangent Function
$y = \arctan x$ or $y = \tan^{-1} x$
Domain $= \{x : x \text{ is a real number}\}$
Range $= \left\{y : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

Note that when we use the TAN⁻¹ key on the graphing calculator with any real number, the response will be an number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

These same restrictions can be used to define inverse functions for the secant, cosecant, and cotangent functions, which will be left to the student. (See Exercises 40–42.)

EXAMPLE 1

Express the range of the arcsine function in degrees.

Solution Expressed in radians, the range of the arcsine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, that is, $-\frac{\pi}{2} \leq \arcsin\theta \leq \frac{\pi}{2}$. Since an angle of $\frac{\pi}{2}$ radians is an angle of 90° and an angle of $-\frac{\pi}{2}$ radians is an angle of $-90°$, the range of the arcsine function is:

$$[-90°, 90°] \quad \text{or} \quad -90° \leq \theta \leq 90° \quad \textit{Answer}$$

EXAMPLE 2

Find $\sin(\cos^{-1}(-0.5))$.

Solution Let $\theta = \cos^{-1}(-0.5)$. Then θ is a second-quadrant angle whose reference angle, r, is $r = \pi - \theta$ in radians or $r = 180 - \theta$ in degrees.

In radians:	In degrees:
$\cos r = 0.5$	$\cos r = 0.5$
$r = \frac{\pi}{3}$	$r = 60°$
$\pi - \theta = \frac{\pi}{3}$	$180° - \theta = 60°$
$\frac{2\pi}{3} = \theta$	$120° = \theta$
$\sin\theta = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$	$\sin\theta = \sin 120° = \frac{\sqrt{3}}{2}$

Alternative Solution To find $\sin(\cos^{-1}(-0.5))$, find $\sin\theta$ when $\theta = \cos^{-1}(-0.5)$. If $\theta = \cos^{-1}(-0.5)$, then $\cos\theta = -0.5$. For the function $y = \cos^{-1}x$, when x is negative, $\frac{\pi}{2} < y \leq \pi$. Therefore, θ is the measure of a second-quadrant angle and $\sin\theta$ is positive.

Use the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + (-0.5)^2 = 1$$
$$\sin^2\theta + 0.25 = 1$$
$$\sin^2\theta = 0.75$$
$$\sin\theta = \sqrt{0.75} = \sqrt{0.25}\sqrt{3} = 0.5\sqrt{3}$$

Check ENTER: SIN 2nd COS⁻¹ (-) .5)) ENTER 2nd √ 3) ÷ 2 ENTER

DISPLAY:

```
sin(cos⁻¹(⁻.5))
      .8660254038
√(3)/2
      .8660254038
```

Answer $\frac{\sqrt{3}}{2}$

EXAMPLE 3

Find the exact value of $\sec\left(\arcsin\frac{-3}{5}\right)$ if the angle is a third-quadrant angle.

Solution Let $\theta = \left(\arcsin\frac{-3}{5}\right)$. This can be written as $\sin\theta = \left(\frac{-3}{5}\right)$. Then:

$$\sec\left(\arcsin\frac{-3}{5}\right) = \sec\theta = \frac{1}{\cos\theta}$$

We know the value of $\sin\theta$. We can use a Pythagorean identity to find $\cos\theta$:

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

$$\frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1-\sin^2\theta}} = \frac{1}{\pm\sqrt{1-\left(-\frac{3}{5}\right)^2}} = \frac{1}{\pm\sqrt{1-\frac{9}{25}}} = \frac{1}{\pm\sqrt{\frac{16}{25}}} = \frac{1}{\pm\frac{4}{5}} = \pm\frac{5}{4}$$

An angle in the third quadrant has a negative secant function value.

Answer $\sec\left(\arcsin\frac{-3}{5}\right) = -\frac{5}{4}$

Calculator Solution

ENTER: 1 **÷** **COS** **2nd** **SIN⁻¹** −3 **÷** 5 **)** **)** **MATH** **ENTER** **ENTER**

DISPLAY:

```
1/cos(sin⁻¹(⁻3/5)
)►Frac
                5/4
```

The calculator returns the value $\frac{5}{4}$. The calculator used the function value of $\left(\arcsin\frac{-3}{5}\right)$, a fourth-quadrant angle for which the secant function value is positive. For the given third-quadrant angle, the secant function value is negative.

Answer $\sec\left(\arcsin\frac{-3}{5}\right) = -\frac{5}{4}$

Exercises

Writing About Mathematics

1. Nicholas said that the restricted domain of the cosine function is the same as the restricted domain of the tangent function. Do you agree with Nicholas? Explain why or why not.

2. Sophia said that the calculator solution to Example 2 could have been found with the calculator set to either degree or radian mode. Do you agree with Sophia? Explain why or why not.

Developing Skills

In 3–14, find each value of θ: **a.** in degrees **b.** in radians

3. $\theta = \arcsin\frac{1}{2}$

4. $\theta = \arctan 1$

5. $\theta = \arccos 1$

6. $\theta = \arctan(-1)$

7. $\theta = \arccos\left(-\frac{1}{2}\right)$

8. $\theta = \arcsin\left(-\frac{1}{2}\right)$

9. $\theta = \arctan\left(-\sqrt{3}\right)$

10. $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$

11. $\theta = \arccos(-1)$

12. $\theta = \arcsin 1$

13. $\theta = \arctan 0$

14. $\theta = \arccos 0$

In 15–23, use a calculator to find each value of θ to the nearest degree.

15. $\theta = \arcsin 0.6$

16. $\theta = \arccos(-0.6)$

17. $\theta = \arctan 4.4$

18. $\theta = \arctan(-4.4)$

19. $\theta = \arccos 0.9$

20. $\theta = \arccos(-0.9)$

21. $\theta = \arcsin 0.72$

22. $\theta = \arcsin(-0.72)$

23. $\theta = \arctan(-17.3)$

In 24–32, find the exact value of each expression.

24. $\sin(\arctan 1)$

25. $\cos(\arctan 0)$

26. $\tan(\arccos 1)$

27. $\cos(\arccos(-1))$

28. $\tan\left(\arcsin\left(-\frac{1}{2}\right)\right)$

29. $\cos\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right)$

30. $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

31. $\sin\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

32. $\cos\left(\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$

In 33–38, find the exact radian measure θ of an angle with the smallest absolute value that satisfies the equation.

33. $\sin\theta = \frac{\sqrt{2}}{2}$

34. $\cos\theta = \frac{\sqrt{3}}{2}$

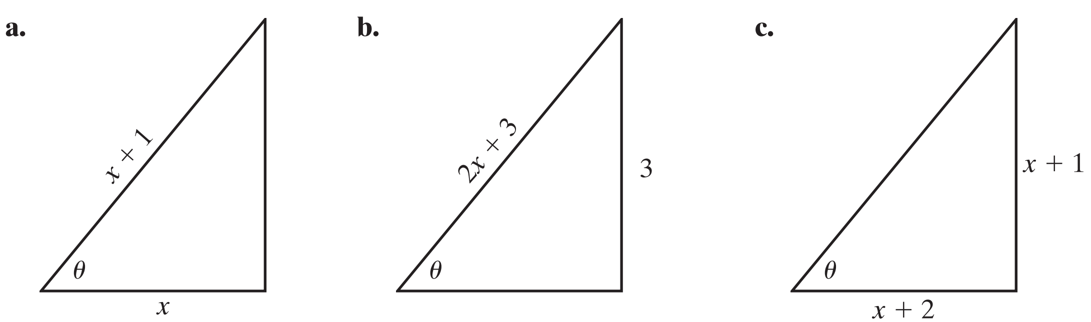
35. $\tan\theta = 1$

36. $\sec\theta = -1$

37. $\csc\theta = -\sqrt{2}$

38. $\cot\theta = \sqrt{3}$

39. In **a–c** with the triangles labeled as shown, use the inverse trigonometric functions to express θ in terms of x.



Applying Skills

40. a. Restrict the domain of the secant function to form a one-to-one function that has an inverse function. Justify your answer.

b. Is the restricted domain found in **a** the same as the restricted domain of the cosine function?

c. Find the range of the restricted secant function.

d. Find the domain of the inverse secant function, that is, the arcsecant function.

e. Find the range of the arcsecant function.

41. a. Restrict the domain of the cosecant function to form a one-to-one function that has an inverse function. Justify your domain.

b. Is the restricted domain found in **a** the same as the restricted domain of the sine function?

c. Find the range of the restricted cosecant function.

d. Find the domain of the inverse cosecant function, that is, the arccosecant function.

e. Find the range of the arccosecant function.

42. a. Restrict the domain of the cotangent function to form a one-to-one function that has an inverse function. Justify your domain.

b. Is the restricted domain found in **a** the same as the restricted domain of the tangent function?

c. Find the range of the restricted cotangent function.

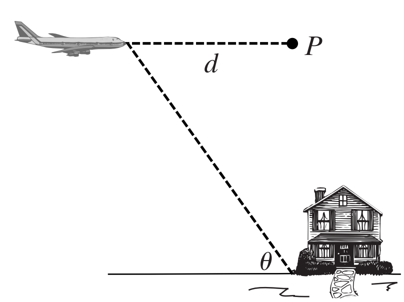
d. Find the domain of the inverse cotangent function, that is, the arccotangent function.

e. Find the range of the arccotangent function.

43. Jennifer lives near the airport. An airplane approaching the airport flies at a constant altitude of 1 mile toward a point, P, above Jennifer's house. Let θ be the measure of the angle of elevation of the plane and d be the horizontal distance from P to the airplane.

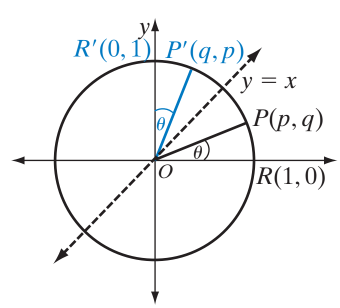
a. Express θ in terms of d.

b. Find θ when $d = 1$ mile and when $d = 0.5$ mile.



10-6 COFUNCTIONS

Two acute angles are complementary if the sum of their measures is 90°. In the diagram, $\angle ROP$ is an angle in standard position whose measure is θ and (p, q) are the coordinates of P, the intersection of $\overrightarrow{OP}$ with the unit circle. Under a reflection in the line $y = x$, the image of (x, y) is (y, x). Therefore, under a reflection in the line $y = x$, the image of $P(p, q)$ is $P'(q, p)$, the image of $R(1, 0)$ is $R'(0, 1)$, and the image of $O(0, 0)$ is $O(0, 0)$. Under a line reflection, angle measure is preserved. Therefore,



$$\text{m}\angle ROP = \text{m}\angle R'OP' = \theta$$
$$\text{m}\angle ROP' = 90° - \text{m}\angle R'OP' = 90° - \theta$$

The two angles, $\angle ROP$ and $\angle ROP'$ are in standard position. Therefore,

$\cos\theta = p$	$\sin\theta = q$
$\sin(90° - \theta) = p$	$\cos(90° - \theta) = q$
$\boldsymbol{\cos\theta = \sin(90° - \theta)}$	$\boldsymbol{\sin\theta = \cos(90° - \theta)}$

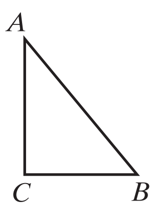
The sine of an acute angle is equal to the cosine of its complement. The sine and cosine functions are called **cofunctions**. There are two other pairs of cofunctions in trigonometry.

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{q}{p}$	$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{p}{q}$
$\cot(90° - \theta) = \frac{\cos(90° - \theta)}{\sin(90° - \theta)} = \frac{q}{p}$	$\tan(90° - \theta) = \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{p}{q}$
$\boldsymbol{\tan\theta = \cot(90° - \theta)}$	$\boldsymbol{\cot\theta = \tan(90° - \theta)}$

The tangent and cotangent functions are cofunctions.

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{p}$	$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{q}$
$\csc(90° - \theta) = \frac{1}{\sin(90° - \theta)} = \frac{1}{p}$	$\sec(90° - \theta) = \frac{1}{\cos(90° - \theta)} = \frac{1}{q}$
$\boldsymbol{\sec\theta = \csc(90° - \theta)}$	$\boldsymbol{\csc\theta = \sec(90° - \theta)}$

The secant and cosecant functions are cofunctions. Cofunctions represented in radians are left to the student. (See Exercise 25.)



In any right triangle ABC, if $\angle C$ is the right angle, then $\angle A$ and $\angle B$ are complementary angles. Therefore, $m\angle A = 90 - m\angle B$ and $m\angle B = 90 - m\angle A$.

- **Sine and cosine are cofunctions: $\sin A = \cos B$ and $\sin B = \cos A$.**
- **Tangent and cotangent are cofunctions: $\tan A = \cot B$ and $\tan B = \cot A$.**
- **Secant and cosecant are cofunctions: $\sec A = \csc B$ and $\sec B = \csc A$.**

EXAMPLE 1

If $\tan 63.44° = 2.00$, find a value of θ such that $\cot\theta = 2.00$.

Solution If θ is the degree measure of a first-quadrant angle, then:

$$\tan(90° - \theta) = \cot\theta$$
$$\tan 63.44° = \cot\theta$$

Therefore, $\tan(90° - \theta) = \tan 63.44$ and $63.44 = (90° - \theta)$.

$$\begin{aligned} 63.44° &= 90° - \theta \\ \theta &= 90° - 63.44° \\ &= 26.56° \end{aligned}$$

Answer $\theta = 26.56°$

EXAMPLE 2

Express sin 100° as the function value of an acute angle that is less than 45°.

Solution An angle whose measure is 100° is a second-quadrant angle whose reference angle is $180° - 100°$ or 80°.

Therefore, $\sin 100° = \sin 80° = \cos (90° - 80°) = \cos 10°$.

Answer $\sin 100° = \cos 10°$

Exercises

Writing About Mathematics

1. Mia said that if you know the sine value of each acute angle, then you can find any trigonometric function value of an angle of any measure. Do you agree with Mia? Explain why or why not.

2. If $\sin A = \frac{1}{2}$, is $\cos A = \frac{\sqrt{3}}{2}$ always true? Explain why or why not.

Developing Skills

In 3–22: **a.** Rewrite each function value in terms of its cofunction. **b.** Find, to four decimal places, the value of the function value found in **a**.

3. sin 65° **4.** cos 80° **5.** tan 54° **6.** sin 86°

7. csc 48° **8.** sec 75° **9.** cot 57° **10.** cos 70°

11. sin 110° **12.** tan 95° **13.** cos 130° **14.** sec 125°

15. sin 230° **16.** cos 255° **17.** tan 237° **18.** csc 266°

19. cos 300° **20.** sin 295° **21.** cot 312° **22.** sec 285°

23. If $\sin \theta = \cos (20 + \theta)$, what is the value of θ?

24. For what value of x does $\tan (x + 10) = \cot (40 + x)$?

25. Complete the following table of cofunctions for radian values.

Cofunctions (degrees)		Cofunctions (radians)	
$\cos \theta = \sin (90° - \theta)$	$\sin \theta = \cos (90° - \theta)$		
$\tan \theta = \cot (90° - \theta)$	$\cot \theta = \tan (90° - \theta)$		
$\sec \theta = \csc (90° - \theta)$	$\csc \theta = \sec (90° - \theta)$		

In 26–33: **a.** Rewrite each function value in terms of its cofunction. **b.** Find the exact value of the function value found in **a**.

26. $\sin \frac{\pi}{3}$ **27.** $\cos \frac{\pi}{4}$ **28.** $\tan \frac{\pi}{6}$

29. $\sec \frac{2\pi}{3}$ **30.** $\csc \frac{5\pi}{6}$ **31.** $\cot \pi$

32. $\sin \left(-\frac{\pi}{4}\right)$ **33.** $\cos \frac{8\pi}{3}$ **34.** $\tan \left(-\frac{5\pi}{3}\right)$

CHAPTER SUMMARY

A **radian** is the unit of measure of a central angle that intercepts an arc equal in length to the radius of the circle.

In general, the radian measure θ of a central angle is the length of the intercepted arc, s, divided by the radius of the circle, r.

$$\theta = \frac{s}{r}$$

The length of the semicircle is πr. Therefore, the radian measure of a straight angle is $\frac{\pi r}{r} = \pi$. This leads to the relationship:

$$\pi \text{ radians} = 180 \text{ degrees}$$

The measure of any angle can be expressed in degrees or in radians. We can do this by forming the following proportion:

$$\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{180}{\pi}$$

When the calculator is in radian mode, it will return the decimal value of the radian measure whose function value is entered. The measure will not be in terms of π.

An **identity** is an equation that is true for all values of the variable for which the terms of the variable are defined.

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\sec \theta = \frac{1}{\cos \theta}$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$	$\cos^2 \theta + \sin^2 \theta = 1$
$\csc \theta = \frac{1}{\sin \theta}$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\cot \theta = \frac{1}{\tan \theta}$		$\cot^2 \theta + 1 = \csc^2 \theta$

The domains and ranges of the six basic trigonometric functions are as follows.

Function	**Domain (n is an integer)**	**Range**
Sine	All real numbers	$[-1, 1]$
Cosine	All real numbers	$[-1, 1]$
Tangent	All real numbers except $\frac{\pi}{2} + n\pi$	All real numbers
Cotangent	All real numbers except $n\pi$	All real numbers
Secant	All real numbers except $\frac{\pi}{2} + n\pi$	$(-\infty, -1] \cup [1, \infty)$
Cosecant	All real numbers except $n\pi$	$(-\infty, -1] \cup [1, \infty)$

The **inverse trigonometric functions** are defined for **restricted domains**.

Sine Function with a Restricted Domain
$y = \sin x$
Domain $= \left\{x : -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$
Range $= \{y : -1 \le y \le 1\}$

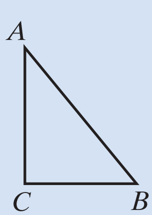
Inverse Sine Function
$y = \arcsin x$ or $y = \sin^{-1} x$
Domain $= \{x : -1 \le x \le 1\}$
Range $= \left\{y : -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}$

Cosine Function with a Restricted Domain
$y = \cos x$
Domain $= \{x : 0 \le x \le \pi\}$
Range $= \{y : -1 \le y \le 1\}$

Inverse Cosine Function
$y = \arccos x$ or $y = \cos^{-1} x$
Domain $= \{x : -1 \le x \le 1\}$
Range $= \{y : 0 \le y \le \pi\}$

Tangent Function with a Restricted Domain
$y = \tan x$
Domain $= \left\{x : -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$
Range $= \{y : y \text{ is a real number}\}$

Inverse Tangent Function
$y = \arctan x$ or $y = \tan^{-1} x$
Domain $= \{x : x \text{ is a real number}\}$
Range $= \left\{y : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$



In any right triangle ABC, if $\angle C$ is the right angle, then $\angle A$ and $\angle B$ are complementary angles. Therefore, $m\angle A = 90 - m\angle B$ and $m\angle B = 90 - m\angle A$.

- Sine and cosine are cofunctions: $\sin A = \cos B$ and $\sin B = \cos A$.
- Tangent and cotangent are cofunctions: $\tan A = \cot B$ and $\tan B = \cot A$.
- Secant and cosecant are cofunctions: $\sec A = \csc B$ and $\sec B = \csc A$.

VOCABULARY

10-1 Radian

10-3 Identity • Pythagorean identity

10-5 Restricted domain • Inverse sine function • Inverse cosine function • Inverse tangent function

10-6 Cofunction

REVIEW EXERCISES

In 1–4, express each degree measure in radian measure in terms of π.

1. 75° **2.** 135° **3.** 225° **4.** −60°

In 5–8, express each radian measure in degree measure.

5. $\frac{\pi}{4}$ **6.** $\frac{2\pi}{5}$ **7.** $\frac{7\pi}{6}$ **8.** $-\frac{\pi}{8}$

9. What is the radian measure of a right angle?

In 10–17, in a circle of radius r, the measure of a central angle is θ and the length of the arc intercepted by the angle is s.

10. If $r = 5$ cm and $s = 5$ cm, find θ.

11. If $r = 5$ cm and $\theta = 2$, find s.

12. If $r = 2$ in. and $s = 3$ in., find θ.

13. If $s = 8$ cm and $\theta = 2$, find r.

14. If $s = 9$ cm and $r = 3$ mm, find θ.

15. If $s = \pi$ cm and $\theta = \frac{\pi}{4}$, find r.

16. If $\theta = 3$ and $s = 7.5$ cm, find r.

17. If $\theta = \frac{\pi}{5}$ and $r = 10$ ft, find s.

18. For what values of θ ($0° \leq \theta \leq 360°$) is each of the following undefined?

a. $\tan\theta$ **b.** $\cot\theta$ **c.** $\sec\theta$ **d.** $\csc\theta$

19. What is the domain and range of each of the following functions?

a. $\arcsin\theta$ **b.** $\arccos\theta$ **c.** $\arctan\theta$

In 20–27, find the exact value of each trigonometric function.

20. $\sin\frac{\pi}{6}$ **21.** $\cos\frac{\pi}{4}$ **22.** $\tan\frac{\pi}{3}$ **23.** $\sin\frac{\pi}{2}$

24. $\cos\frac{2\pi}{3}$ **25.** $\tan\frac{3\pi}{4}$ **26.** $\sin\frac{11\pi}{6}$ **27.** $\cos\left(\frac{-7\pi}{6}\right)$

In 28–33, find the exact value of each expression.

28. $\tan \frac{3\pi}{4} + \cot \frac{3\pi}{4}$

29. $2 \csc \frac{\pi}{2} + 5 \cot \frac{\pi}{2}$

30. $2 \tan \frac{2\pi}{3} - 3 \sec \frac{5\pi}{6}$

31. $\tan^2\left(\frac{7\pi}{6}\right) - \sec^2\left(\frac{7\pi}{6}\right)$

32. $\cot^2\left(\frac{\pi}{3}\right) + \sec^2\left(\frac{\pi}{4}\right)$

33. $3 \cot\left(\frac{\pi}{4}\right) \cdot 2 \csc^2\left(\frac{\pi}{4}\right)$

In 34–39, find the exact value of each expression.

34. $\sin\left(\arcsin -\frac{\sqrt{3}}{2}\right)$

35. $\tan\left(\text{arccot } -\sqrt{3}\right)$

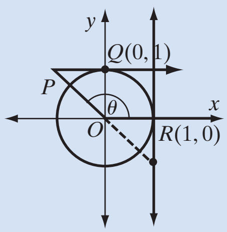
36. $\cot(\arccos -1)$

37. $\sec\left(\arcsin \frac{\sqrt{2}}{2}\right)$

38. $\csc(\text{arccot } -1)$

39. $\cos(\text{arcsec } 2)$

Exploration



In this exploration, we will continue the work begun in Chapter 9.

Let $\overrightarrow{OP}$ intersect the unit circle at P. The line that is tangent to the circle at $R(1, 0)$ intersects $\overleftrightarrow{OP}$ at T. The line that is tangent to the circle at $Q(0, 1)$ intersects $\overleftrightarrow{OP}$ at S. The measure of $\angle ROP$ is θ.

1. $\angle ROP$ is a second-quadrant angle as shown to the left. Locate points S and T. Show that $OT = \sec \theta$, $OS = \csc \theta$, and $QS = \cot \theta$.

2. Draw $\angle ROP$ as a third-quadrant angle. Let $R(1, 0)$ and $Q(0, 1)$ be fixed points. Locate points S and T. Show that $OT = \sec \theta$, $OS = \csc \theta$, and $QS = \cot \theta$.

3. Repeat step 2 for a fourth-quadrant angle.

CUMULATIVE REVIEW CHAPTERS 1–10

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If $\sec \theta = \sqrt{5}$, then $\cos \theta$ equals

(1) $-\sqrt{5}$ (2) $\frac{2\sqrt{5}}{5}$ (3) $\frac{\sqrt{5}}{5}$ (4) $-\frac{\sqrt{5}}{5}$

2. Which of the following is an irrational number?

(1) $0.\overline{34}$ (2) $\sqrt{121}$ (3) $3 + 2i$ (4) 2π

3. Which of the following is a function from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ that is one-to-one and onto?
(1) $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 5)\}$ (3) $\{(1, 3), (2, 3), (3, 3), (4, 3), (5, 3)\}$
(2) $\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$ (4) $\{(1, 1), (2, 4), (3, 1), (4, 2), (5, 1)\}$

4. When written with a rational denominator, $\frac{1}{3 + \sqrt{5}}$ is equal to
(1) $\frac{3 + \sqrt{5}}{14}$ (2) $\frac{3 - \sqrt{5}}{-2}$ (3) $\frac{3 - \sqrt{5}}{4}$ (4) $\frac{3 + \sqrt{5}}{8}$

5. The fraction $\frac{a - \frac{1}{a}}{1 - \frac{1}{a^2}}$ is equal to
(1) $\frac{a^2}{a - 1}$ (2) $\frac{a}{a - 1}$ (3) $\frac{a^2}{a + 1}$ (4) $\frac{a}{a + 1}$

6. The inverse function of the function $y = 2x + 1$ is
(1) $y = \frac{1}{2}x + 1$ (3) $y = \frac{x - 1}{2}$
(2) $y = \frac{x + 1}{2}$ (4) $y = -2x - 1$

7. The exact value of $\cos \frac{2\pi}{3}$ is
(1) $\frac{\sqrt{3}}{2}$ (2) $-\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

8. The series $-2 + 2 + 6 + 10 + \cdots$ written in sigma notation is
(1) $\sum_{k=1}^{\infty} -2k$ (3) $\sum_{k=1}^{\infty} (k + 4k)$
(2) $\sum_{k=1}^{\infty} (-2 + 2k)$ (4) $\sum_{k=1}^{\infty} (-2 + 4(k - 1))$

9. What are the zeros of the function $y = x^3 + 3x^2 + 2x$?
(1) 0 only (3) 0, 1, and 2
(2) -2, -1, and 0 (4) The function has no zeros.

10. The product $\left(\sqrt{-8}\right)\left(\sqrt{-2}\right)$ is equal to
(1) $4i$ (2) 4 (3) -4 (4) $3i\sqrt{2}$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Express the roots of $x^2 - 6x + 13 = 0$ in $a + bi$ form.

12. If 280 is the measure of an angle in degrees, what is the measure of the angle in radians?

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. What is the equation of a circle if the endpoints of the diameter are $(-2, 4)$ and $(6, 0)$?

14. Given $\log_{\frac{1}{3}} \frac{1}{9} - 2 \log_{\frac{1}{3}} \frac{1}{27} + \log_{\frac{1}{3}} \frac{1}{243}$:

a. Write the expression as a single logarithm.

b. Evaluate the logarithm.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. If $\sec \theta = \sqrt{3}$ and θ is in the fourth quadrant, find $\cos \theta$, $\sin \theta$, and $\tan \theta$ in simplest radical form.

16. Find graphically the common solutions of the equations:

$$y = x^2 - 4x + 1$$
$$y = x - 3$$