CHAPTER

5

QUADRATIC FUNCTIONS AND COMPLEX NUMBERS

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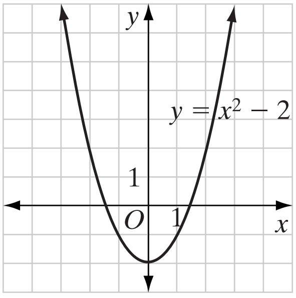
Records of early Babylonian and Chinese mathematics provide methods for the solution of a quadratic equation. These methods are usually presented in terms of directions for the solution of a particular problem, often one derived from the relationship between the perimeter and the area of a rectangle. For example, a list of steps needed to find the length and width of a rectangle would be given in terms of the perimeter and the area of a rectangle.

The Arab mathematician Mohammed ibn Musa al-Khwarizmi gave similar steps for the solution of the relationship "a square and ten roots of the same amount to thirty-nine dirhems." In algebraic terms, x is the root, x^2 is the square, and the equation can be written as $x^2 + 10x = 39$. His solution used these steps:

1. Take half of the number of roots : $\frac{1}{2}$ of $10 = 5$
2. Multiply this number by itself: $5 \times 5 = 25$
3. Add this to 39: $25 + 39 = 64$
4. Now take the root of this number: $\sqrt{64} = 8$
5. Subtract from this number one-half the number of roots: $8 - \frac{1}{2}$ of $10 = 8 - 5 = 3$
6. This is the number which we sought.

In this chapter, we will derive a formula for the solution of any quadratic equation. The derivation of this formula uses steps very similar to those used by al-Khwarizmi.

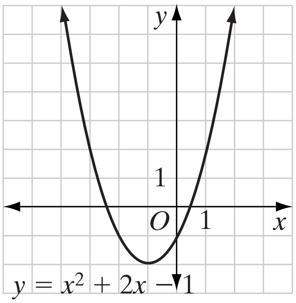
5-1 REAL ROOTS OF A QUADRATIC EQUATION



The real roots of a polynomial function are the x-coordinates of the points at which the graph of the function intersects the x-axis. The graph of the polynomial function $y = x^2 - 2$ is shown at the right. The graph intersects the x-axis between -2 and -1 and between 1 and 2. The roots of $0 = x^2 - 2$ appear to be about -1.4 and 1.4. Can we find the exact values of these roots? Although we cannot factor $x^2 - 2$ over the set of integers, we can find the value of x^2 and take the square roots of each side of the equation.

$$0 = x^2 - 2$$
$$2 = x^2$$
$$\pm\sqrt{2} = x$$

The roots of the function $y = x^2 - 2$ are the irrational numbers $-\sqrt{2}$ and $\sqrt{2}$.



The graph of the polynomial function $y = x^2 + 2x - 1$ is shown at the right. The graph shows that the function has two real roots, one between -3 and -2 and the other between 0 and 1. Can we find the exact values of the roots of the function $y = x^2 + 2x - 1$? We know that the equation $0 = x^2 + 2x - 1$ cannot be solved by factoring the right member over the set of integers. However, we can follow a process similar to that used to solve $0 = x^2 - 2$. We need to write the right member as a trinomial that is the square of a binomial. This process is called **completing the square**.

Recall that $(x + h)^2 = x^2 + 2hx + h^2$. To write a trinomial that is a perfect square using the terms $x^2 + 2x$, we need to let $2h = 2$. Therefore, $h = 1$ and $h^2 = 1$. The perfect square trinomial is $x^2 + 2x + 1$, which is the square of $(x + 1)$.

(1) Add 1 to both sides of the equation to isolate the terms in x: $\quad 0 = x^2 + 2x - 1$, $\quad 1 = x^2 + 2x$

(2) Add the number needed to complete the square to both sides of the equation: $\quad 2 = x^2 + 2x + 1$, $\quad 2 = (x + 1)^2$

$$\left(\tfrac{1}{2}\text{ coefficient of } x\right)^2 = \left(\tfrac{1}{2}(2)\right)^2 = 1^2 = 1$$

(3) Write the square root of both sides of the equation: $\quad \pm\sqrt{2} = x + 1$

(4) Solve for x: $\quad -1 \pm \sqrt{2} = x$

The roots of $y = x^2 + 2x - 1$ are $-1 - \sqrt{2}$ and $-1 + \sqrt{2}$.

We can use a calculator to approximate these roots to the nearest hundredth.

ENTER: (-) 1 − 2nd √ 2 ENTER

DISPLAY:
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-1-√(2
        -2.414213562
```

ENTER: (-) 1 + 2nd √ 2 ENTER

DISPLAY:
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-1+√(2
        .4142135624
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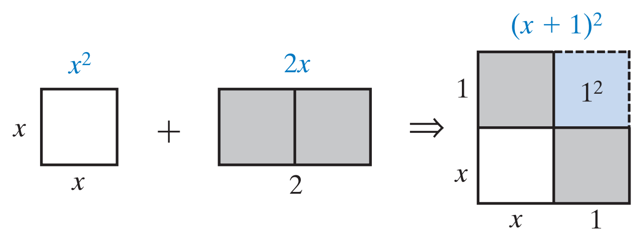
To the nearest hundredth, the roots are -2.41 and 0.41. One root is between -3 and -2 and the other is between 0 and 1, as shown on the graph.

The process of completing the square can be illustrated geometrically. Let us draw the quadratic expression $x^2 + 2x$ as a square with sides of length x and a rectangle with dimensions $x \times 2$. Cut the rectangle in half and attach the two parts to the sides of the square. The result is a larger square with a missing corner. The factor $\left(\frac{1}{2}\text{ coefficient of } x\right)^2$ or 1^2 is the area of this missing corner. This is the origin of the expression "completing the square."



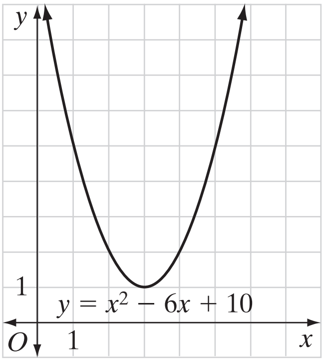
Procedure

To solve a quadratic equation of the form $ax^2 + bx - c = 0$ where $a = 1$ by completing the square:

1. Isolate the terms in x on one side of the equation.
2. Add the square of one-half the coefficient of x or $\left(\frac{1}{2}b\right)^2$ to both sides of the equation.
3. Write the square root of both sides of the resulting equation and solve for x.

Note: The procedure outlined above only works when $a = 1$. If the coefficient a is not equal to 1, divide each term by a, and then proceed with completing the square using the procedure outlined above.

Not every second-degree polynomial function has real roots. Note that the graph of $y = x^2 - 6x + 10$ does not intersect the x-axis and therefore has no real roots. Can we find roots by completing the square?



Use $\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(-6)\right)^2 = (-3)^2 = 9$ to complete the square.

$$\begin{aligned} 0 &= x^2 - 6x + 10 \\ -10 &= x^2 - 6x && \text{Isolate the terms in } x. \\ -10 + 9 &= x^2 - 6x + 9 && \text{Add the square of one-half the} \\ -1 &= (x - 3)^2 && \text{coefficient of } x \text{ to complete square.} \\ \pm\sqrt{-1} &= x - 3 && \text{Take the square root of both sides.} \\ 3 \pm \sqrt{-1} &= x && \text{Solve for } x. \end{aligned}$$

Since there is no real number that is the square root of -1, this function has no real zeros. We will study a set of numbers that includes the square roots of negative numbers later in this chapter.

EXAMPLE 1

Find the exact values of the zeros of the function $f(x) = -x^2 - 4x + 2$.

Solution *How to Proceed*

(1) Let $f(x) = 0$: $\qquad 0 = -x^2 - 4x + 2$

(2) Isolate the terms in x: $\qquad -2 = -x^2 - 4x$

(3) Multiply by -1: $\qquad 2 = x^2 + 4x$

(4) Complete the square by adding $\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(4)\right)^2 = 2^2 = 4$ to both sides of the equation:

$$\begin{aligned} 4 + 2 &= x^2 + 4x + 4 \\ 6 &= x^2 + 4x + 4 \\ 6 &= (x + 2)^2 \end{aligned}$$

(5) Take the square root of each side of the equation: $\qquad \pm\sqrt{6} = x + 2$

(6) Solve for x: $\qquad -2 \pm \sqrt{6} = x$

Answer The zeros are $(-2 + \sqrt{6})$ and $(-2 - \sqrt{6})$.

EXAMPLE 2

Find the roots of the function $y = 2x^2 - x - 1$.

Solution *How to Proceed*

(1) Let $y = 0$: $\quad 0 = 2x^2 - x - 1$

(2) Isolate the terms in x: $\quad 1 = 2x^2 - x$

(3) Divide both sides of the equation by the coefficient of x^2: $\quad \frac{1}{2} = x^2 - \frac{1}{2}x$

(4) Complete the square by adding

$\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2} \cdot -\frac{1}{2}\right)^2 = \frac{1}{16}$

to both sides of the equation:

$$\frac{1}{16} + \frac{1}{2} = x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2$$

$$\frac{1}{16} + \frac{8}{16} = \left(x - \frac{1}{4}\right)^2$$

$$\frac{9}{16} = \left(x - \frac{1}{4}\right)^2$$

(5) Take the square root of both sides of the equation: $\quad \pm\sqrt{\frac{9}{16}} = x - \frac{1}{4}$

(6) Solve for x: $\quad \frac{1}{4} \pm \frac{3}{4} = x$

The roots of the function are:

$$\frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = -\frac{1}{2} \qquad \text{and} \qquad \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$$

Answer $-\frac{1}{2}$ and 1

Note that since the roots are rational, the equation could have been solved by factoring over the set of integers.

$$0 = 2x^2 - x - 1$$

$$0 = (2x + 1)(x - 1)$$

$$\begin{array}{r|l} 2x + 1 = 0 & x - 1 = 0 \\ 2x = -1 & x = 1 \\ x = -\frac{1}{2} & \end{array}$$

Graphs of Quadratic Functions and Completing the Square

A quadratic function $y = a(x - h)^2 + k$ can be graphed in terms of the basic quadratic function $y = x^2$. In particular, the graph of $y = a(x - h)^2 + k$ is the graph of $y = x^2$ shifted h units horizontally and k units vertically. The coefficient a determines its orientation: when $a > 0$, the parabola opens upward; when

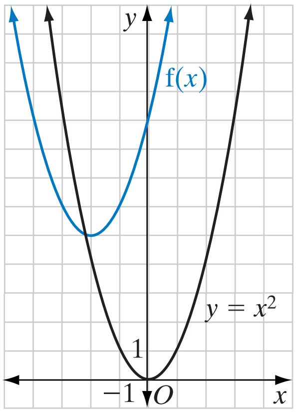
$a < 0$, the parabola opens downward. The coefficient a also determines how narrow or wide the graph is compared to $y = x^2$: when $|a| > 1$, the graph is narrower; when $|a| < 1$, the graph is wider.

We can use the process of completing the square to help us graph quadratic functions of the form $y = ax^2 + bx + c$, as the following examples demonstrate:

EXAMPLE 3

Graph $f(x) = x^2 + 4x + 9$.

Solution The square of one-half the coefficient of x is $\left(\frac{1}{2} \cdot 4\right)^2$ or 4.



Add *and* subtract 4 to keep the function equivalent to the original and complete the square.

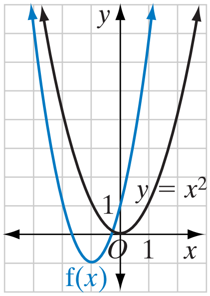
$$\begin{aligned} f(x) &= x^2 + 4x + 9 \\ &= x^2 + 4x + 4 - 4 + 9 \\ &= (x^2 + 4x + 4) + 5 \\ &= (x + 2)^2 + 5 \end{aligned}$$

The graph of $f(x)$ is the graph of $y = x^2$ shifted 2 units to the left and 5 units up, as shown on the right. Note that this function has no real roots.

EXAMPLE 4

Graph $f(x) = 2x^2 + 4x + 1$.

Solution *How to Proceed*



(1) Factor 2 from the first two terms: $f(x) = 2x^2 + 4x + 1$

$= 2(x^2 + 2x) + 1$

(2) Add *and* subtract $\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2} \cdot 2\right)^2 = 1$ to keep the function equivalent to the original:

$= 2(x^2 + 2x + 1 - 1) + 1$

$= 2(x^2 + 2x + 1) - 2 + 1$

$= 2(x^2 + 2x + 1) - 1$

(3) Complete the square: $= 2(x + 1)^2 - 1$

The graph of $f(x)$ is the graph of $y = x^2$ stretched vertically by a factor of 2, and shifted 1 unit to the left and 1 unit down as shown on the left. Note that this function has two real roots, one between -2 and -1 and the other between -1 and 0.

Exercises

Writing About Mathematics

1. To solve the equation given in step 2 of Example 2, Sarah multiplied each side of the equation by 8 and then added 1 to each side to complete the square. Show that $16x^2 - 8x - 8$ is the square of a binomial and will lead to the correct solution of $0 = 2x^2 - x - 1$.

2. Phillip said that the equation $0 = x^2 - 6x + 1$ can be solved by adding 8 to both sides of the equation. Do you agree with Phillip? Explain why or why not.

Developing Skills

In 3–8, complete the square of the quadratic expression.

3. $x^2 + 6x$ **4.** $x^2 - 8x$ **5.** $x^2 - 2x$

6. $x^2 - 12x$ **7.** $2x^2 - 4x$ **8.** $x^2 - 3x$

In 9–14: **a.** Sketch the graph of each function. **b.** From the graph, estimate the roots of the function to the nearest tenth. **c.** Find the exact irrational roots in simplest radical form.

9. $f(x) = x^2 - 6x + 4$ **10.** $f(x) = x^2 - 2x - 2$ **11.** $f(x) = x^2 + 4x + 2$

12. $f(x) = x^2 - 6x + 6$ **13.** $f(x) = x^2 - 2x - 1$ **14.** $f(x) = x^2 - 10x + 18$

In 9–26, solve each quadratic equation by completing the square. Express the answer in simplest radical form.

15. $x^2 - 2x - 2 = 0$ **16.** $x^2 + 6x + 4 = 0$ **17.** $x^2 - 4x + 1 = 0$

18. $x^2 + 2x - 5 = 0$ **19.** $x^2 - 6x + 2 = 0$ **20.** $x^2 - 8x + 4 = 0$

21. $2x^2 + 12x + 3 = 0$ **22.** $3x^2 - 6x - 1 = 0$ **23.** $2x^2 - 6x + 3 = 0$

24. $4x^2 - 20x + 9 = 0$ **25.** $\frac{1}{2}x^2 + x - 3 = 0$ **26.** $\frac{1}{4}x^2 + \frac{3}{4}x - \frac{3}{2} = 0$

27. a. Complete the square to find the roots of the equation $x^2 - 5x + 1 = 0$.

b. Write, to the nearest tenth, a rational approximation for the roots.

In 28–33, without graphing the parabola, describe the translation, reflection, and/or scaling that must be applied to $y = x^2$ to obtain the graph of each given function.

28. $f(x) = x^2 - 12x + 5$ **29.** $f(x) = x^2 + 2x - 2$ **30.** $f(x) = x^2 - 6x - 7$

31. $f(x) = x^2 + x + \frac{9}{4}$ **32.** $f(x) = -x^2 + x + 2$ **33.** $f(x) = 3x^2 + 6x + 3$

34. Determine the coordinates of the vertex and the equation of the axis of symmetry of $f(x) = x^2 + 8x + 5$ by writing the equation in the form $f(x) = (x - h)^2 + k$. Justify your answer.

Applying Skills

35. The length of a rectangle is 4 feet more than twice the width. The area of the rectangle is 38 square feet.

a. Find the dimensions of the rectangle in simplest radical form.

b. Show that the product of the length and width is equal to the area.

c. Write, to the nearest tenth, rational approximations for the length and width.

36. One base of a trapezoid is 8 feet longer than the other base and the height of the trapezoid is equal to the length of the shorter base. The area of the trapezoid is 20 square feet.

a. Find the lengths of the bases and of the height of the trapezoid in simplest radical form.

b. Show that the area of the trapezoid is equal to one-half the height times the sum of the lengths of the bases.

c. Write, to the nearest tenth, rational approximations for the lengths of the bases and for the height of the trapezoid.

37. Steve and Alice realized that their ages are consecutive odd integers. The product of their ages is 195. Steve is younger than Alice. Determine their ages by completing the square.

5-2 THE QUADRATIC FORMULA

We have solved equations by using the same steps to complete the square needed to find the roots or zeros of a function of the form $y = ax^2 + bx + c$, that is, to solve the quadratic equation $0 = ax^2 + bx + c$. We will apply those steps to the general equation to find a formula that can be used to solve any quadratic equation.

How to Proceed

(1) Let $y = 0$:

$$0 = ax^2 + bx + c$$

(2) Isolate the terms in x:

$$-c = ax^2 + bx$$

(3) Divide both sides of the equation by the coefficient of x^2:

$$\frac{-c}{a} = x^2 + \frac{b}{a}x$$

(4) Complete the square by adding $\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ to both sides of the equation:

$$\frac{b^2}{4a^2} + \frac{-c}{a} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

$$\frac{b^2}{4a^2} + \frac{-4ac}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

(5) Take the square root of each side of the equation:

$$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$$

(6) Solve for x:

$$\frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x$$

This result is called the **quadratic formula**.

▶ **When $a \neq 0$, the roots of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.**

EXAMPLE 1

Use the quadratic formula to find the roots of $2x^2 - 4x = 1$.

Solution *How to Proceed*

(1) Write the equation in standard form:

$$2x^2 - 4x = 1$$
$$2x^2 - 4x - 1 = 0$$

(2) Determine the values of a, b, and c: $a = 2, b = -4, c = -1$

(3) Substitute the values of a, b, and c in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)}$$

(4) Perform the computation:

$$x = \frac{4 \pm \sqrt{16 + 8}}{4}$$
$$x = \frac{4 \pm \sqrt{24}}{4}$$

(5) Write the radical in simplest form:

$$x = \frac{4 \pm \sqrt{4}\sqrt{6}}{4}$$
$$x = \frac{4 \pm 2\sqrt{6}}{4}$$

(6) Divide numerator and denominator by 2:

$$x = \frac{2 \pm \sqrt{6}}{2}$$

Answer $\frac{2 + \sqrt{6}}{2}$ and $\frac{2 - \sqrt{6}}{2}$

EXAMPLE 2

Use the quadratic formula to show that the equation $x^2 + x + 2 = 0$ has no real roots.

Solution For the equation $x^2 + x + 2 = 0$, $a = 1$, $b = 1$, and $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{1 - 8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

There is no real number that is the square root of a negative number. Therefore, $\sqrt{-7}$ is not a real number and the equation has no real roots.

EXAMPLE 3

One leg of a right triangle is 1 centimeter shorter than the other leg and the hypotenuse is 2 centimeters longer than the longer leg. What are lengths of the sides of the triangle?

Solution Let $x =$ the length of the longer leg

$x - 1 =$ the length of the shorter leg

$x + 2 =$ the length of the hypotenuse

Use the Pythagorean Theorem to write an equation.

$$x^2 + (x - 1)^2 = (x + 2)^2$$
$$x^2 + x^2 - 2x + 1 = x^2 + 4x + 4$$
$$x^2 - 6x - 3 = 0$$

Use the quadratic formula to solve this equation: $a = 1, b = -6, c = -3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-3)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36 + 12}}{2}$$
$$= \frac{6 \pm \sqrt{48}}{2} = \frac{6 \pm \sqrt{16}\sqrt{3}}{2} = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3}$$

The roots are $3 + 2\sqrt{3}$ and $3 - 2\sqrt{3}$. Reject the negative root, $3 - 2\sqrt{3}$.

The length of the longer leg is $3 + 2\sqrt{3}$.

The length of the shorter leg is $3 + 2\sqrt{3} - 1$ or $2 + 2\sqrt{3}$.

The length of the hypotenuse is $3 + 2\sqrt{3} + 2$ or $5 + 2\sqrt{3}$.

Check

$$(3 + 2\sqrt{3})^2 + (2 + 2\sqrt{3})^2 \stackrel{?}{=} (5 + 2\sqrt{3})^2$$
$$(9 + 12\sqrt{3} + 4 \cdot 3) + (4 + 8\sqrt{3} + 4 \cdot 3) \stackrel{?}{=} 25 + 20\sqrt{3} + 4 \cdot 3$$
$$37 + 20\sqrt{3} = 37 + 20\sqrt{3} \checkmark$$

Exercises

Writing About Mathematics

1. Noah said that the solutions of Example 1, $\frac{2 + \sqrt{6}}{2}$ and $\frac{2 - \sqrt{6}}{2}$, could have been written as $\frac{\cancelto{1}{2} + \sqrt{6}}{\cancelto{1}{2}} = 1 + \sqrt{6}$ and $\frac{\cancelto{1}{2} - \sqrt{6}}{\cancelto{1}{2}} = 1 - \sqrt{6}$. Do you agree with Noah? Explain why or why not.

2. Rita said that when a, b, and c are real numbers, the roots of $ax^2 + bx + c = 0$ are real numbers only when $b^2 \geq 4ac$. Do you agree with Rita? Explain why or why not.

Developing Skills

In 3–14, use the quadratic formula to find the roots of each equation. Irrational roots should be written in simplest radical form.

3. $x^2 + 5x + 4 = 0$	**4.** $x^2 + 6x - 7 = 0$	**5.** $x^2 - 3x + 1 = 0$
6. $x^2 - x - 4 = 0$	**7.** $x^2 + 5x - 2 = 0$	**8.** $x^2 - 8 = 0$
9. $x^2 - 3x = 0$	**10.** $x^2 + 2x = 4$	**11.** $3x^2 - 5x + 2 = 0$
12. $4x^2 - x - 1 = 0$	**13.** $5x + 1 = 2x^2$	**14.** $2x^2 = x + 4$
15. $x^2 - 6x + 3 = 0$	**16.** $4x^2 - 4x = 11$	**17.** $3x^2 = 4x + 2$

18. **a.** Sketch the graph of $y = x^2 + 6x + 2$.

b. From the graph, estimate the roots of the function to the nearest tenth.

c. Use the quadratic formula to find the exact values of the roots of the function.

d. Express the roots of the function to the nearest tenth and compare these values to your estimate from the graph.

Applying Skills

In 19–25, express each answer in simplest radical form. Check each answer.

19. The larger of two numbers is 5 more than twice the smaller. The square of the smaller is equal to the larger. Find the numbers.

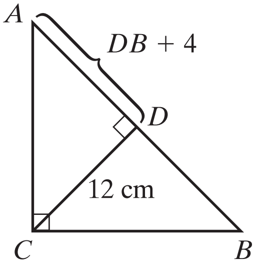
20. The length of a rectangle is 2 feet more than the width. The area of the rectangle is 2 square feet. What are the dimensions of the rectangle?

21. The length of a rectangle is 4 centimeters more than the width. The measure of a diagonal is 10 centimeters. Find the dimensions of the rectangle.

22. The length of the base of a triangle is 6 feet more than the length of the altitude to the base. The area of the triangle is 18 square feet. Find the length of the base and of the altitude to the base.

23. The lengths of the bases of a trapezoid are $x + 10$ and $3x + 2$ and the length of the altitude is $2x$. If the area of the trapezoid is 40, find the lengths of the bases and of the altitude.

24. The altitude, $\overline{CD}$, to the hypotenuse, $\overline{AB}$, of right triangle ABC separates the hypotenuse into two segments, $\overline{AD}$ and $\overline{DB}$. If $AD = DB + 4$ and $CD = 12$ centimeters, find DB, AD, and AB. Recall that the length of the altitude to the hypotenuse of a right triangle is the mean proportional between the lengths of the segments into which the hypotenuse is separated, that is, $\frac{AD}{CD} = \frac{CD}{DB}$.



25. A parabola is symmetric under a line reflection. Each real root of the quadratic function $y = ax^2 + bx + c$ is the image of the other under a reflection in the axis of symmetry of the parabola.

a. What are the coordinates of the points at which the parabola whose equation is $y = ax^2 + bx + c$ intersects the x-axis?

b. What are the coordinates of the midpoint of the points whose coordinates were found in part **a**?

c. What is the equation of the axis of symmetry of the parabola $y = ax^2 + bx + c$?

d. The turning point of a parabola is on the axis of symmetry. What is the x-coordinate of the turning point of the parabola $y = ax^2 + bx + c$?

26. Gravity on the moon is about one-sixth of gravity on Earth. An astronaut standing on a tower 20 feet above the moon's surface throws a ball upward with a velocity of 30 feet per second. The height of the ball at any time t (in seconds) is $\text{h}(t) = -2.67t^2 + 30t + 20$. To the nearest tenth of a second, how long will it take for the ball to hit the ground?

27. Based on data from a local college, a statistician determined that the average student's grade point average (GPA) at this college is a function of the number of hours, h, he or she studies each week. The grade point average can be estimated by the function $\text{G}(h) = 0.006h^2 + 0.02h + 1.2$ for $0 \leq h \leq 20$.

a. To the nearest tenth, what is the average GPA of a student who does no studying?

b. To the nearest tenth, what is the average GPA of a student who studies 12 hours per week?

c. To the nearest tenth, how many hours of studying are required for the average student to achieve a 3.2 GPA?

28. Follow the steps given in the chapter opener to find a root of the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Compare your solution with the quadratic formula.

Hands-On Activity: Alternate Derivation of the Quadratic Formula

As you may have noticed, the derivation of the quadratic formula on page 193 uses fractions throughout (in particular, starting with step 3). We can use an alternate derivation of the quadratic equation that uses fractions only in the last step. This alternate derivation uses the following perfect square trinomial:

$$\begin{aligned} 4a^2x^2 + 4abx + b^2 &= 4a^2x^2 + 2abx + 2abx + b^2 \\ &= 2ax(2ax + b) + b(2ax + b) \\ &= (2ax + b)(2ax + b) \\ &= (2ax + b)^2 \end{aligned}$$

To write $ax^2 + bx$, the terms in x of the general quadratic equation, as this perfect square trinomial, we must multiply by $4a$ and add b^2. The steps for the derivation of the quadratic formula using this perfect square trinomial are shown below.

How to Proceed

(1) Isolate the terms in x: $$ax^2 + bx + c = 0$$ $$ax^2 + bx = -c$$

(2) Multiply both sides of the equation by $4a$: $$4a^2x^2 + 4abx = -4ac$$

(3) Add b^2 to both sides of the equation to complete the square: $$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$ $$(2ax + b)^2 = b^2 - 4ac$$

(4) Now solve for x.

Does this procedure lead to the quadratic formula?

5-3 THE DISCRIMINANT

When a quadratic equation $ax^2 + bx + c = 0$ has rational numbers as the values of a, b, and c, the value of $b^2 - 4ac$, or the **discriminant**, determines the nature of the roots. Consider the following equations.

CASE 1 *The discriminant $b^2 - 4ac$ is a positive perfect square.*

$2x^2 + 3x - 2 = 0$ where $a = 2$, $b = 3$, and $c = -2$.

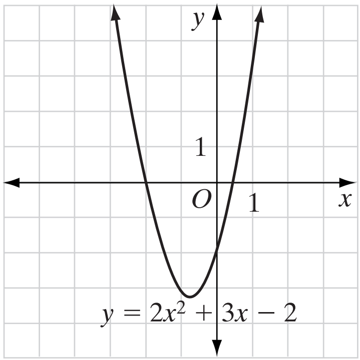
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4}$$

The roots are $\frac{-3 + 5}{4} = \frac{2}{4} = \frac{1}{2}$ and $\frac{-3 - 5}{4} = \frac{-8}{4} = -2$.

The roots are unequal rational numbers, and the graph of the corresponding quadratic function has two x-intercepts.



CASE 2 *The discriminant $b^2 - 4ac$ is a positive number that is not a perfect square.*

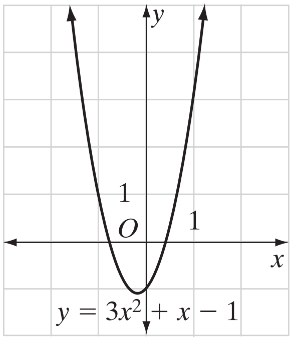
$3x^2 + x - 1 = 0$ where $a = 3$, $b = 1$, and $c = -1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{-1 \pm \sqrt{1 + 12}}{6} = \frac{-1 \pm \sqrt{13}}{6}$$

The roots are $\frac{-1 + \sqrt{13}}{6}$ and $\frac{-1 - \sqrt{13}}{6}$.



The roots are unequal irrational numbers, and the graph of the corresponding quadratic function has two x-intercepts.

CASE 3 *The discriminant $b^2 - 4ac$ is 0.*

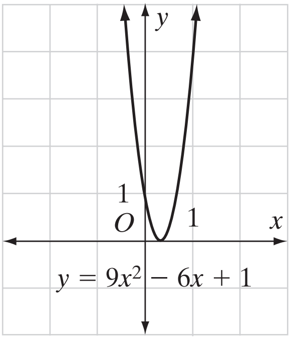
$9x^2 - 6x + 1 = 0$ where $a = 9$, $b = -6$, and $c = 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(1)}}{2(9)}$$

$$= \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6 \pm \sqrt{0}}{18} = \frac{6 \pm 0}{18}$$

The roots are $\frac{6 + 0}{18} = \frac{1}{3}$ and $\frac{6 - 0}{18} = \frac{1}{3}$.



The roots are equal rational numbers, and the graph of the corresponding quadratic function has one x-intercept. The root is a **double root**.

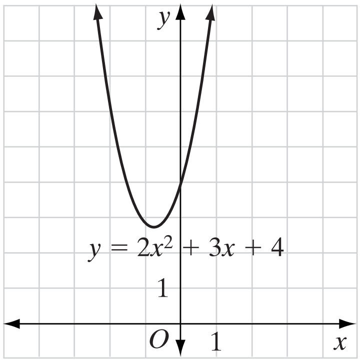
CASE 4 *The discriminant $b^2 - 4ac$ is a negative number.*

$2x^2 + 3x + 4 = 0$ where $a = 2$, $b = 3$, and $c = 4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(4)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9 - 32}}{4} = \frac{-3 \pm \sqrt{-23}}{4}$$



This equation has no real roots because there is no real number equal to $\sqrt{-23}$, and the graph of the corresponding quadratic function has no x-intercepts.

In each case, when a, b, and c are rational numbers, the value of $b^2 - 4ac$ determined the nature of the roots and the x-intercepts. From the four cases given above, we can make the following observations.

Let $ax^2 + bx + c = 0$ be a quadratic equation and a, b, and c be rational numbers with $a \neq 0$:

When the discriminant $b^2 - 4ac$ is:	The roots of the equation are:	The number of x-intercepts of the function is:
>0 and a perfect square	real, rational, and unequal	2
>0 and not a perfect square	real, irrational, and unequal	2
$=0$	real, rational, and equal	1
<0	not real numbers	0

EXAMPLE 1

Without solving the equation, find the nature of the roots of $3x^2 + 2x = 1$.

Solution Write the equation in standard form: $3x^2 + 2x - 1 = 0$.

Evaluate the discriminant of the equation when $a = 3, b = 2, c = -1$.

$$b^2 - 4ac = 2^2 - 4(3)(-1) = 4 + 12 = 16$$

Since a, b, and c are rational and the discriminant is greater than 0 and a perfect square, the roots are real, rational, and unequal.

EXAMPLE 2

Show that it is impossible to draw a rectangle whose area is 20 square feet and whose perimeter is 10 feet.

Solution Let l and w be the length and width of the rectangle.

How to Proceed

(1) Use the formula for perimeter: $2l + 2w = 10$

(2) Solve for l in terms of w:
$$l + w = 5$$
$$l = 5 - w$$

(3) Use the formula for area: $lw = 20$

(4) Substitute $5 - w$ for l. Write the resulting quadratic equation in standard form:
$$\begin{aligned}(5 - w)(w) &= 20\\ 5w - w^2 &= 20\\ -w^2 + 5w - 20 &= 0\\ w^2 - 5w + 20 &= 0\end{aligned}$$

(5) Test the discriminant:
$$\begin{aligned}b^2 - 4ac &= (-5)^2 - 4(1)(20)\\ &= 25 - 80\\ &= -55\end{aligned}$$

Since the discriminant is negative, the quadratic equation has no real roots. There are no real numbers that can be the dimensions of this rectangle.

EXAMPLE 3

A lamp manufacturer earns a weekly profit of P dollars according to the function $P(x) = -0.3x^2 + 50x - 170$ where x is the number of lamps sold. When $P(x) = 0$, the roots of the equation represent the level of sales for which the manufacturer will break even.

a. Interpret the meaning of the discriminant value of 2,296.

b. Use the discriminant to determine the number of break even points.

c. Use the discriminant to determine if there is a level of sales for which the weekly profit would be $3,000.

Solution **a.** Since the discriminant is greater than 0 and not a perfect square, the equation has two roots or break even points that are irrational and unequal. *Answer*

b. 2 (see part **a**) *Answer*

c. We want to know if the equation

$$3{,}000 = -0.3x^2 + 50x - 170$$

has at least one real root. Find the discriminant of this equation. The equation in standard form is $0 = -0.3x^2 + 50x - 3{,}170$.

$$b^2 - ac = 50^2 - 4(-0.3)(-3{,}170) = -1{,}304$$

The discriminant is negative, so there are no real roots. The company cannot achieve a weekly profit of $3,000. *Answer*

Exercises

Writing About Mathematics

1. a. What is the discriminant of the equation $x^2 + (\sqrt{5})x - 1 = 0$?

b. Find the roots of the equation $x^2 + (\sqrt{5})x - 1 = 0$.

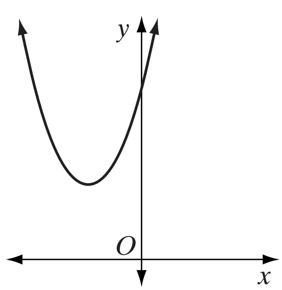
c. Do the rules for the relationship between the discriminant and the roots of the equation given in this chapter apply to this equation? Explain why or why not.

2. Christina said that when a, b, and c are rational numbers and a and c have opposite signs, the quadratic equation $ax^2 + bx + c = 0$ must have real roots. Do you agree with Christina? Explain why or why not.

Developing Skills

In 3–8, each graph represents a quadratic function. Determine if the discriminant of the quadratic function is greater than 0, equal to 0, or less than 0.

3.



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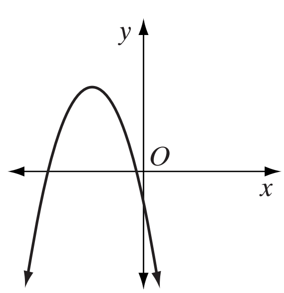

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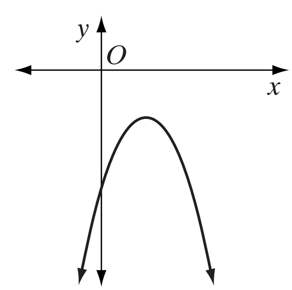

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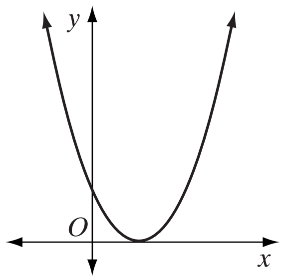

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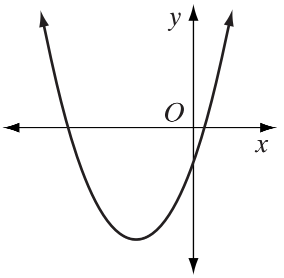

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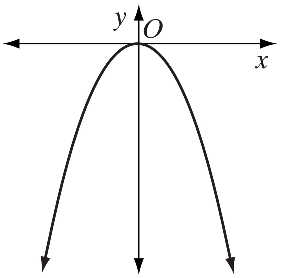


In 9–14: **a.** For each given value of the discriminant of a quadratic equation with rational coefficients, determine if the roots of the quadratic equation are (1) rational and unequal, (2) rational and equal, (3) irrational and unequal, or (4) not real numbers. **b.** Use your answer to part **a** to determine the number of x-intercepts of the graph of the corresponding quadratic function.

9. 9 **10.** 5 **11.** 0 **12.** 12 **13.** -4 **14.** 49

In 15–23: **a.** Find the value of the discriminant and determine if the roots of the quadratic equation are (1) rational and unequal, (2) rational and equal, (3) irrational and unequal, or (4) not real numbers. **b.** Use any method to find the real roots of the equation if they exist.

15. $x^2 - 12x + 36 = 0$ **16.** $2x^2 + 7x = 0$ **17.** $x^2 + 3x + 1 = 0$

18. $2x^2 - 8 = 0$ **19.** $4x^2 - x = 1$ **20.** $3x^2 = 5x - 3$

21. $4x - 1 = 4x^2$ **22.** $2x^2 - 3x - 5 = 0$ **23.** $3x - x^2 = 5$

24. When $b^2 - 4ac = 0$, is $ax^2 + bx + c$ a perfect square trinomial or a constant times a perfect square trinomial? Explain why or why not.

25. Find a value of c such that the roots of $x^2 + 2x + c = 0$ are:

a. equal and rational. **b.** unequal and rational.

c. unequal and irrational. **d.** not real numbers.

26. Find a value of b such that the roots of $x^2 + bx + 4 = 0$ are:

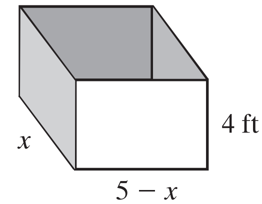
a. equal and rational. **b.** unequal and rational.

c. unequal and irrational. **d.** not real numbers.

Applying Skills

27. Lauren wants to fence off a rectangular flower bed with a perimeter of 30 yards and a diagonal length of 8 yards. Use the discriminant to determine if her fence can be constructed. If possible, determine the dimensions of the rectangle.

28. An open box is to be constructed as shown in the figure on the right. Is it possible to construct a box with a volume of 25 cubic feet? Use the discriminant to explain your answer.



29. The height, in feet, to which of a golf ball rises when shot upward from ground level is described by the function $h(t) = -16t^2 + 48t$ where t is the time elapsed in seconds. Use the discriminant to determine if the golf ball will reach a height of 32 feet.

30. The profit function for a company that manufactures cameras is $P(x) = -x^2 + 350x - 15{,}000$. Under present conditions, can the company achieve a profit of \$20,000? Use the discriminant to explain your answer.

5-4 THE COMPLEX NUMBERS

The Set of Imaginary Numbers

We have seen that when we solve quadratic equations with rational coefficients, some equations have real, rational roots, some have real, irrational roots, and some do not have real roots.

For example:

$x^2 - 9 = 0$	$x^2 - 8 = 0$	$x^2 + 9 = 0$
$x^2 = 9$	$x^2 = 8$	$x^2 = -9$
$x = \pm\sqrt{9}$	$x = \pm\sqrt{8}$	$x = \pm\sqrt{-9}$
$x = \pm 3$	$x = \pm\sqrt{4}\sqrt{2}$	$x = \pm\sqrt{9}\sqrt{-1}$
	$x = \pm 2\sqrt{2}$	$x = \pm 3\sqrt{-1}$

The equations $x^2 - 9 = 0$ and $x^2 - 8 = 0$ have real roots, but the equation $x^2 + 9 = 0$ does not have real roots.

A number is an abstract idea to which we assign a meaning and a symbol. We can form a new set of numbers that includes $\sqrt{-1}$, to which we will assign the symbol ***i***. We call i the unit of the **set of imaginary numbers**.

The numbers $+3\sqrt{-1}$ and $-3\sqrt{-1}$ that are the roots of the equation $x^2 + 9 = 0$ can be written as $3i$ and $-3i$.

DEFINITION

A number of the form $a\sqrt{-1} = ai$ is a **pure imaginary number** where a is a non-zero real number.

The idea of the square root of -1 as a number was not accepted by many early mathematicians. Rafael Bombelli (1526–1572) introduced the terms *plus of minus* and *minus of minus* to designate these numbers. For instance, he designated $2 + 3i$ as 2 *plus of minus* 3 and $2 - 3i$ as 2 *minus of minus* 3. The term *imaginary* was first used in 1637 by René Descartes, who used it in an almost derogatory sense, implying that somehow imaginary numbers had no significance in mathematics. However, the algebra of imaginary numbers was later developed and became an invaluable tool in real-world applications such as alternating current.

Some elements of the set of pure imaginary numbers are:

$$\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i$$
$$-\sqrt{-16} = -\sqrt{4}\sqrt{-1} = -2i$$
$$\sqrt{-10} = \sqrt{10}\sqrt{-1} = \sqrt{-1}\sqrt{10} = i\sqrt{10}$$
$$-\sqrt{-32} = -\sqrt{16}\sqrt{2}\sqrt{-1} = -4\sqrt{-1}\sqrt{2} = -4i\sqrt{2}$$

Note the order in which the factors of $i\sqrt{10}$ and $-4i\sqrt{2}$ are written. The factor that is a radical is written last to avoid confusion with $\sqrt{10i}$ and $-4\sqrt{2i}$.

EXAMPLE 1

Simplify: **a.** $\sqrt{-7}$ **b.** $\sqrt{-12}$

Solution Write each imaginary number in terms of i and simplify.

a. $\sqrt{-7} = \sqrt{7}\sqrt{-1} = i\sqrt{7}$ **b.** $\sqrt{-12} = \sqrt{4}\sqrt{3}\sqrt{-1} = 2i\sqrt{3}$

Powers of i

The unit element of the set of pure imaginary numbers is $i = \sqrt{-1}$. Since $\sqrt{b}$ is one of the two equal factors of b, then $i = \sqrt{-1}$ is one of the two equal factors of -1 and $i \cdot i = -1$ or $i^2 = -1$. The box at the right shows the first four powers of i.

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = i^2 \cdot i = -1i = -i$
$i^4 = i^2 \cdot i^2 = -1(-1) = 1$

In general, for any integer n:

$$\mathbf{i^{4n} = (i^4)^n = 1^n = 1}$$

$$\mathbf{i^{4n+1} = (i^4)^n(i^1) = 1^n(i) = 1(i) = i}$$

$$\mathbf{i^{4n+2} = (i^4)^n(i^2) = 1^n(-1) = 1(-1) = -1}$$

$$\mathbf{i^{4n+3} = (i^4)^n(i^3) = 1^n(-i) = 1(-i) = -i}$$

For any positive integer k, i^k is equal to i, -1, $-i$, or 1. We can use this relationship to evaluate any power of i.

EXAMPLE 2

Evaluate: *Answers*

a. i^{14} $= i^{4(3)+2} = (i^4)^3 \cdot (i^2) = (1)^3(-1) = 1(-1) = -1$

b. i^7 $= i^7 = i^{4+3} = i^4 \cdot i^3 = (1)(-i) = -i$

Arithmetic of Imaginary Numbers

The rule for multiplying radicals, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, is true if and only if a and b are non-negative numbers. When $a < 0$ or $b < 0$, $\sqrt{a} \cdot \sqrt{b} \neq \sqrt{ab}$. For example, it is *incorrect* to write: $\sqrt{-4} \times \sqrt{-9} = \sqrt{36} = 6$, but it is *correct* to write:

$$\sqrt{-4} \times \sqrt{-9} = \sqrt{4}\sqrt{-1} \times \sqrt{9}\sqrt{-1} = 2i \times 3i = 6i^2 = 6(-1) = -6$$

The distributive property of multiplication over addition is true for imaginary numbers.

$$\begin{aligned} &\sqrt{-4} + \sqrt{-9} \\ &= \sqrt{4}\sqrt{-1} + \sqrt{9}\sqrt{-1} \\ &= 2i + 3i \\ &= (2 + 3)i \\ &= 5i \end{aligned} \qquad \begin{aligned} &\sqrt{-8} + \sqrt{-50} \\ &= \sqrt{4}\sqrt{-1}\sqrt{2} + \sqrt{25}\sqrt{-1}\sqrt{2} \\ &= 2i\sqrt{2} + 5i\sqrt{2} \\ &= (2 + 5)\left(i\sqrt{2}\right) \\ &= 7i\sqrt{2} \end{aligned}$$

EXAMPLE 3

Find the sum of $\sqrt{-25} + \sqrt{-49}$.

Solution Write each imaginary number in terms of i and add similar terms.

$$\sqrt{-25} + \sqrt{-49} = \sqrt{25}\sqrt{-1} + \sqrt{49}\sqrt{-1} = 5i + 7i = 12i \text{ Answer}$$

The Set of Complex Numbers

The set of pure imaginary numbers is not closed under multiplication because the powers of i include both real numbers and imaginary numbers. For example, $i \cdot i = -1$. We want to define a set of numbers that includes both real numbers and pure imaginary numbers and that satisfies all the properties of the real numbers. This new set of numbers is called the set of *complex numbers* and is defined as the set of numbers of the form $a + bi$ where a and b are real numbers. When $a = 0$, $a + bi = 0 + bi = bi$, a pure imaginary number. When $b = 0$, $a + bi = a + 0i = a$, a real number. Therefore, both the set of pure imaginary numbers and the set of real numbers are contained in the set of complex numbers. A complex number that is *not* a real number is called an imaginary number.

DEFINITION

A **complex number** is a number of the form

$$a + bi$$

where a and b are real numbers and $i = \sqrt{-1}$.

Examples of complex numbers are:

$$2, \quad -4, \quad \tfrac{1}{3}, \quad \sqrt{2}, \quad \pi, \quad 4i, \quad -5i, \quad \tfrac{1}{4}i, \quad i\sqrt{5}, \quad \pi i,$$
$$2 - 3i, \quad -7 + i, \quad 8 + 3i, \quad 0 - i, \quad 5 + 0i$$

Complex Numbers and the Graphing Calculator

The TI family of graphing calculators can use complex numbers. To change your calculator to work with complex numbers, press MODE, scroll down to the row that lists REAL, select $a + bi$, and then press ENTER. Your calculator will now evaluate complex numbers. For example, with your calculator in $a + bi$ mode, write $6 + \sqrt{-9}$ as a complex number in terms of i.

ENTER: 6 + 2nd √ −9) ENTER

DISPLAY:

```
6+√(-9)
                6+3i
```

EXAMPLE 4

Evaluate each expression using a calculator:

a. $\sqrt{-25} - \sqrt{-16}$ **b.** $2 + \sqrt{-9} + 5\sqrt{-4}$

Calculator Solution

a. Press 2nd √ −25) − 2nd √ −16) ENTER.

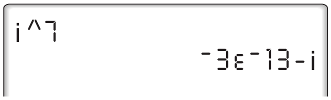
```
√(-25)-√(-16)
                   i
```

b. Press 2 + 2nd √ −9) + 5 2nd √ −4) ENTER.

```
2+√(-9)+5√(-4)
                2+13i
```

Answers **a.** i **b.** $2 + 13i$

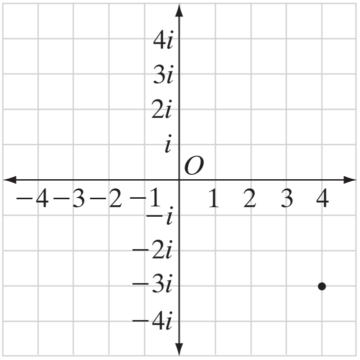
What about powers of i? A calculator is not an efficient tool for evaluating an expression such as bi^n for large values of n. Large complex powers introduce rounding errors in the answer as shown in the calculator screen on the right.



Graphing Complex Numbers

We are familiar with the one-to-one correspondence between the set of real numbers and the points on the number line. The complex numbers can be put in a one-to-one correspondence with the points of a plane. This correspondence makes it possible for a physical quantity, such as an alternating current that has both amplitude and direction, to be represented by a complex number.

On graph paper, draw a horizontal real number line. Through 0 on the real number line, draw a vertical line. This vertical line is the pure imaginary number line. On this line, mark points equidistant from one another such that the distance between them is equal to the unit distance on the real number line. Label the points above 0 as i, $2i$, $3i$, $4i$, and so on and the numbers below 0 as $-i$, $-2i$, $-3i$, $-4i$, and so on.



To find the point that corresponds to the complex number $4 - 3i$, move to 4 on the real number line and then, parallel to the imaginary number line, move down 3 units to a point on the same horizontal line as $3i$.

Hands-On Activity

On the complex plane, locate the points that correspond to $4 + 2i$ and $2 - 5i$. Draw a line segment from 0 to each of these points. Now draw a parallelogram with these two line segments as adjacent sides. What is the complex number that corresponds to the vertex of the parallelogram opposite 0? How is this complex number related to $4 + 2i$ and $2 - 5i$? Repeat the procedure given above for each of the following pairs of complex numbers.

1. $2 + 4i, 3 + i$ **2.** $-2 + 5i, -5 + 2i$ **3.** $4 - 2i, 2 - 2i$

4. $-4 - 2i, 3 - 5i$ **5.** $1 + 4i, 1 - 4i$ **6.** $-5 + 3i, -5 - 3i$

7. $-3 + 0i, 0 - 4i$ **8.** $4 + 0i, -4 + 3i$ **9.** $4, 2i$

Exercises

Writing About Mathematics

1. Pete said that $\sqrt{-2} \times \sqrt{-8} = \sqrt{16} = 4$. Do you agree with Pete? Explain why or why not.

2. Ethan said that the square of any pure imaginary number is a negative real number. Do you agree with Ethan? Justify your answer.

Developing Skills

In 3–18, write each number in terms of i.

3. $\sqrt{-4}$ **4.** $\sqrt{-81}$ **5.** $\sqrt{-9}$ **6.** $-\sqrt{-36}$

7. $-\sqrt{-121}$ **8.** $\sqrt{-8}$ **9.** $\sqrt{-12}$ **10.** $-\sqrt{-72}$

11. $5\sqrt{-27}$ **12.** $-\frac{1}{2}\sqrt{-80}$ **13.** $-\sqrt{-51}$ **14.** $\sqrt{-500}$

15. $5 + \sqrt{-5}$ **16.** $1 + \sqrt{-3}$ **17.** $-4 - \sqrt{-24}$ **18.** $-3 + 2\sqrt{-9}$

In 19–34, write each sum or difference in terms of i.

19. $\sqrt{-100} + \sqrt{-81}$ **20.** $\sqrt{-25} - \sqrt{-4}$

21. $\sqrt{-144} + \sqrt{-1}$ **22.** $\sqrt{-49} - \sqrt{-16}$

23. $\sqrt{-12} + \sqrt{-27} - \sqrt{-75}$ **24.** $2\sqrt{-5} + \sqrt{-125}$

25. $\sqrt{4} + \sqrt{-4} + \sqrt{-36} - \sqrt{36}$ **26.** $\sqrt{50} + \sqrt{-2} + \sqrt{200} + \sqrt{-50}$

27. $\sqrt{4} + \sqrt{-32} + \sqrt{-8}$ **28.** $3 + \sqrt{-28} - 7 - \sqrt{-7}$

29. $-3 - \sqrt{-10} + 2 - \sqrt{-90}$ **30.** $-2 + \sqrt{-16} + 7 - \sqrt{-49}$

31. $3 + \sqrt{-36} - 6 + \sqrt{-1}$ **32.** $\frac{1}{7} + \sqrt{-\frac{1}{8}} + \frac{2}{7} - \sqrt{-\frac{1}{2}}$

33. $-\frac{1}{2} + \sqrt{-\frac{2}{3}} - \frac{1}{2} + \sqrt{-\frac{24}{9}}$ **34.** $\sqrt{0.2025} + \sqrt{-0.09}$

In 35–43, write each number in simplest form.

35. $3i + 2i^3$ **36.** $5i^2 + 2i^4$ **37.** $i - 5i^3$

38. $2i^5 + 7i^7$ **39.** $i + i^3 + i^5$ **40.** $4i + 5i^8 + 6i^3 + 2i^4$

41. $i^2 + i^{12} + i^8$ **42.** $i + i^2 + i^3 + i^4$ **43.** i^{57}

In 44–51, locate the point that corresponds to each of the given complex numbers.

44. $2 + 4i$ **45.** $-2 + 5i$ **46.** $4 - 2i$ **47.** $-4 - 2i$

48. $\frac{1}{2} + 4i$ **49.** $0 + 3i$ **50.** $-3 + 0i$ **51.** $0 + 0i$

Applying Skills

52. Imaginary numbers are often used in electrical engineering. The *impedance* Z (measured in ohms) of a circuit measures the resistance of a circuit to alternating current (AC) electricity. For two AC circuits connected in series, the total impedance is the sum $Z_1 + Z_2$ of the individual impedances. Find the total impedance if $Z_1 = 5i$ ohms and $Z_2 = 8i$ ohms.

53. In certain circuits, the total impedance Z_T is given by the formula $Z_T = \frac{Z_1Z_2}{Z_1 + Z_2}$. Find Z_T when $Z_1 = -3i$ and $Z_2 = 4i$.

Hands-On Activity: Multiplying Complex Numbers

Multiplication by i

In the complex plane, the image of 1 under a rotation of 90° about the origin is i, which is equivalent to multiplying 1 by i. Now rotate i by 90° about the origin. The image of i is -1, which models the product $i \cdot i = -1$. This result is true for *any* complex number. That is, multiplying any complex number $a + bi$ by i is equivalent to rotating that number by 90° about the origin. Use this graphical representation of multiplication by i to evaluate the following products:

1. $(2 + 3i) \cdot i$ **2.** $12i \cdot i$ **3.** $\left(\frac{1}{2} + \frac{3}{2}i\right) \cdot i$

4. $-5i \cdot i$ **5.** $-10 \cdot i$ **6.** $(-3 - 2i) \cdot i$

Multiplication by a real number

In the complex plane, the image of any number a under a dilation of k is ak, which is equivalent to multiplying a by k. The image any imaginary number ai under a dilation of k is aki, which is equivalent to multiplying ai by k. This result is true for *any* complex number. That is, multiplying any complex number $a + bi$ by a real number k is equivalent to a dilation of k. Use this graphical representation of multiplication by a real number to evaluate the following products:

1. $(2 + 4i) \cdot 4$ **2.** $(-5 - 5i) \cdot \frac{1}{2}$ **3.** $\left(\frac{3}{4} + 2i\right) \cdot 4$

4. $2i \cdot 2$ **5.** $-4i \cdot \frac{1}{2}$ **6.** $(-3 - 2i) \cdot 3i = (-3 - 2i) \cdot 3 \cdot i$

5-5 OPERATIONS WITH COMPLEX NUMBERS

Like the real numbers, the complex numbers are closed, commutative, and associative under addition and multiplication, and multiplication is distributive over addition.

Adding Complex Numbers

We can add complex numbers using the commutative, associative, and distributive properties that we use to add real numbers. For example:

$$\begin{aligned}(4 + 3i) + (2 + 5i) &= 4 + (3i + 2) + 5i && \text{Associative Property}\\ &= 4 + (2 + 3i) + 5i && \text{Commutative Property}\\ &= (4 + 2) + (3i + 5i) && \text{Associative Property}\\ &= (4 + 2) + (3 + 5)i && \text{Distributive Property}\\ &= 6 + 8i && \text{Substitution Property}\end{aligned}$$

When we know that each of these steps leads to a correct sum, we can write:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Here are some other examples of the addition of complex numbers:

$$(8 - i) + (3 - 7i) = (8 + 3) + (-1 - 7)i = 11 - 8i$$
$$(3 + 2i) + (1 - 2i) = (3 + 1) + (2 - 2)i = 4 + 0i = 4$$
$$(-9 + i) + (9 + 6i) = (-9 + 9) + (1 + 6)i = 0 + 7i = 7i$$

Note that the sum of two complex numbers is always a complex number. That sum may be an element of one of the subsets of the complex numbers, that is, a real number or a pure imaginary number.

Since for any complex number $a + bi$:

$$(a + bi) + (0 + 0i) = (0 + 0i) + (a + bi) = (a + bi)$$

the real number $(0 + 0i)$ or 0 is the additive identity.

► **The additive identity of the complex numbers is the real number $(0 + 0i)$ or 0.**

Similarly, since for any complex number $a + bi$:

$$(a + bi) + (-a - bi) = [a + (-a)] + [b + (-b)]i = 0 + 0i = 0$$

the additive inverse of any complex number $(a + bi)$ is $(-a - bi)$.

► **The additive inverse of any complex number $(a + bi)$ is $(-a - bi)$.**

On some calculators, the key for i is the 2nd function of the decimal point key. When the calculator is in $a + bi$ mode, we can add complex numbers.

ENTER: 4 [+] 2 [2nd] [*i*] [+] 3 [−] 5 [2nd] [*i*] [ENTER]

DISPLAY:

```
4+2i+3-5i
              7-3i
```

EXAMPLE 1

Express the sum in $a + bi$ form: $(-3 + 7i) + (-5 + 2i)$

Solution $(-3 + 7i) + (-5 + 2i) = (-3 - 5) + (7 + 2)i = -8 + 9i$

Calculator Solution ENTER: (-3 + 7 2nd i) + (-5 +
2 2nd i) ENTER

DISPLAY:

```
(⁻3+7i)+(⁻5+2i)
                 ⁻8+9i
```

Answer $-8 + 9i$

Subtracting Complex Numbers

In the set of real numbers, $a - b = a + (-b)$, that is, $a - b$ is the sum of a and the additive inverse of b. We can apply the same rule to the subtraction of complex numbers.

EXAMPLE 2

Subtract:	*Answers*
a. $(5 + 3i) - (2 + 8i)$	$= (5 + 3i) + (-2 - 8i) = (5 - 2) + (3 - 8)i$ $= 3 - 5i$
b. $(7 + 4i) - (-1 + 3i)$	$= (7 + 4i) + (1 - 3i) = (7 + 1) + (4 - 3)i$ $= 8 + 1i = 8 + i$
c. $(6 + i) - (3 + i)$	$= (6 + i) + (-3 - i) = (6 - 3) + (1 - 1)i$ $= 3 + 0i = 3$
d. $(1 + 2i) - (1 - 9i)$	$= (1 + 2i) + (-1 + 9i) = (1 - 1) + (2 + 9)i$ $= 0 + 11i = 11i$

Multiplying Complex Numbers

We can multiply complex numbers using the same principles that we use to multiply binomials. When simplifying each of the following products, recall that $i(i) = i^2 = -1$.

EXAMPLE 3

Multiply:

a. $(5 + 2i)(3 + 4i)$ **b.** $(3 - 5i)(2 + i)$

c. $(9 + 3i)(9 - 3i)$ **d.** $(2 + 6i)^2$

Solution **a.**
$$\begin{aligned} &(5 + 2i)(3 + 4i)\\ &= 5(3 + 4i) + 2i(3 + 4i)\\ &= 15 + 20i + 6i + 8i^2\\ &= 15 + 26i + 8(-1)\\ &= 15 + 26i - 8\\ &= 7 + 26i \quad \textit{Answer} \end{aligned}$$

b.
$$\begin{aligned} &(3 - 5i)(2 + i)\\ &= 3(2 + i) - 5i(2 + i)\\ &= 6 + 3i - 10i - 5i^2\\ &= 6 - 7i - 5(-1)\\ &= 6 - 7i + 5\\ &= 11 - 7i \quad \textit{Answer} \end{aligned}$$

c.
$$\begin{aligned} &(9 + 3i)(9 - 3i)\\ &= 9(9 - 3i) + 3i(9 - 3i)\\ &= 81 - 27i + 27i - 9i^2\\ &= 81 + 0i - 9(-1)\\ &= 81 + 9\\ &= 90 \quad \textit{Answer} \end{aligned}$$

d.
$$\begin{aligned} &(2 + 6i)^2\\ &= (2 + 6i)(2 + 6i)\\ &= 2(2 + 6i) + 6i(2 + 6i)\\ &= 4 + 12i + 12i + 36i^2\\ &= 4 + 24i + 36(-1)\\ &= -32 + 24i \quad \textit{Answer} \end{aligned}$$

In part **c** of Example 3, the product is a real number.

Since for any complex number $a + bi$:

$$(a + bi)(1 + 0i) = a(1 + 0i) + bi(1 + 0i) = a + 0i + bi + 0i = a + bi$$

the real number $1 + 0i$ or 1 is the multiplicative identity.

► **The multiplicative identity of the complex numbers is the real number $1 + 0i$ or 1.**

Does every non-zero complex number have a multiplicative inverse? In the set of rational numbers, the multiplicative inverse of a is $\frac{1}{a}$. We can say that in the set of complex numbers, the multiplicative inverse of $a + bi$ is $\frac{1}{a + bi}$. To express $\frac{1}{a + bi}$ in the form of a complex number, multiply numerator and denominator by the **complex conjugate** of $a + bi$, $a - bi$. The reason for doing so is that the product of a complex number and its conjugate is a real number. For example,

$$\begin{aligned} (1 + 2i)(1 - 2i) &= 1(1 - 2i) + 2i(1 - 2i)\\ &= 1 - 2i + 2i - 4i^2\\ &= 1 - 4(-1)\\ &= 5 \end{aligned}$$

In general,

$$\begin{aligned}(a + bi)(a - bi) &= a(a - bi) + bi(a - bi)\\ &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

or

$$\mathbf{(a + bi)(a - bi) = a^2 + b^2}$$

The product of a complex number $a + bi$ and its conjugate $a - bi$ is a real number, $a^2 + b^2$. Thus, if we the multiply numerator and denominator of $\frac{1}{a + bi}$ by the complex conjugate of the denominator, we will have an equivalent fraction with a denominator that is a real number.

$$\frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

For example, the complex conjugate of $2 - 4i$ is $2 + 4i$ and the multiplicative inverse of $2 - 4i$ is

$$\begin{aligned}\frac{1}{2 - 4i} = \frac{1}{2 - 4i} \cdot \frac{2 + 4i}{2 - 4i} = \frac{2 + 4i}{2^2 + 4^2} &= \frac{2}{2^2 + 4^2} + \frac{4i}{2^2 + 4^2}\\ &= \frac{2}{20} + \frac{4}{20}i\\ &= \frac{1}{10} + \frac{1}{5}i\end{aligned}$$

We can check that $\frac{1}{10} + \frac{1}{5}i$ is the multiplicative inverse of $2 - 4i$ by multiplying the two numbers together. If $\frac{1}{10} + \frac{1}{5}i$ is indeed the multiplicative inverse, then their product will be 1.

$$\begin{aligned}(2 - 4i) \cdot \left(\tfrac{1}{10} + \tfrac{1}{5}i\right) &\stackrel{?}{=} 1\\ 2\left(\tfrac{1}{10} + \tfrac{1}{5}i\right) - 4i\left(\tfrac{1}{10} + \tfrac{1}{5}i\right) &\stackrel{?}{=} 1\\ 2\left(\tfrac{1}{10}\right) + 2\left(\tfrac{1}{5}i\right) - 4i\left(\tfrac{1}{10}\right) - 4i\left(\tfrac{1}{5}i\right) &\stackrel{?}{=} 1\\ \tfrac{2}{10} + \tfrac{2}{5}i - \tfrac{4}{10}i - \tfrac{4}{5}i^2 &\stackrel{?}{=} 1\\ \tfrac{1}{5} + \tfrac{2}{5}i - \tfrac{2}{5}i - \tfrac{4}{5}(-1) &\stackrel{?}{=} 1\\ \tfrac{1}{5} + \tfrac{4}{5} &= 1 \checkmark\end{aligned}$$

► **For any non-zero complex number $a + bi$, its multiplicative inverse is**

$$\frac{1}{a + bi} \quad \textbf{or} \quad \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

EXAMPLE 4

Express the product in $a + bi$ form: $\left(\frac{1}{3} - \frac{1}{2}i\right)(12 + 3i)$

Solution

$$
\begin{aligned}
\left(\tfrac{1}{3} - \tfrac{1}{2}i\right)(12 + 3i) &= \tfrac{1}{3}(12 + 3i) - \tfrac{1}{2}i(12 + 3i) \\
&= 4 + i - 6i - \tfrac{3}{2}i^2 \\
&= 4 - 5i - \tfrac{3}{2}(-1) \\
&= \tfrac{8}{2} + \tfrac{3}{2} - 5i \\
&= \tfrac{11}{2} - 5i
\end{aligned}
$$

Calculator Solution

ENTER: [( ] 1 [÷] 3 [−] 1 [÷] 2 [2nd] [i] [ )] [( ] 12 [+] 3 [2nd] [i] [ )] [MATH] [ENTER] [ENTER]

DISPLAY:

```
(1/3-1/2i)(12+3i
)►Frac
                11/2-5i
```

Answer $\frac{11}{2} - 5i$

Dividing Complex Numbers

We can write $(2 + 3i) \div (1 + 2i)$ as $\frac{2 + 3i}{1 + 2i}$. However, this is not a complex number of the form $a + bi$. Can we write $\frac{2 + 3i}{1 + 2i}$ as an equivalent fraction with a rational denominator? In Chapter 3, we found a way to write a fraction with an irrational denominator as an equivalent fraction with a rational denominator. We can use a similar procedure to write a fraction with a complex denominator as an equivalent fraction with a real denominator.

To write the quotient $\frac{2 + 3i}{1 + 2i}$ with a denominator that is a real number, multiply numerator and denominator of the fraction by the complex conjugate of the denominator. The complex conjugate of $1 + 2i$ is $1 - 2i$.

$$
\begin{aligned}
\frac{2 + 3i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{2 - 4i + 3i - 6i^2}{1^2 - (2i)^2} &= \frac{2 - i - 6i^2}{1 - 4i^2} \\
&= \frac{2 - i - 6(-1)}{1 - 4(-1)} \\
&= \frac{8 - i}{5} \\
&= \frac{8}{5} - \frac{1}{5}i
\end{aligned}
$$

The quotient $(2 + 3i) \div (1 + 2i) = \frac{2 + 3i}{1 + 2i} = \frac{8}{5} - \frac{1}{5}i$ is a complex number in $a + bi$ form.

EXAMPLE 5

Express the quotient in $a + bi$ form: $\frac{5 + 10i}{\frac{1}{2} - \frac{2}{3}i}$

Solution *How to Proceed*

(1) Multiply numerator and denominator by 6 to express the denominator in terms of integers:

$$\frac{5 + 10i}{\frac{1}{2} - \frac{2}{3}i} = \frac{5 + 10i}{\frac{1}{2} - \frac{2}{3}i} \cdot \frac{6}{6} = \frac{30 + 60i}{3 - 4i}$$

(2) Multiply numerator and denominator of the fraction by the complex conjugate of the denominator, $3 + 4i$:

$$\frac{30 + 60i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} = \frac{90 + 120i + 180i + 240i^2}{3^2 + 4^2}$$

(3) Replace i^2 by its equal, -1:

$$= \frac{90 + 120i + 180i + 240(-1)}{9 + 16}$$

(4) Combine like terms:

$$= \frac{90 + 120i + 180i - 240}{25}$$

$$= \frac{-150 + 300i}{25}$$

(5) Divide numerator and denominator by 25:

$$= -6 + 12i$$ *Answer*

Calculator Solution

ENTER: (5 + 10 2nd i) ÷ (1 ÷ 2 − 2 ÷ 3 2nd i) ENTER

DISPLAY:
```
(5+10i)/(1/2-2/3
i)
              ⁻6+12i
```

Exercises

Writing About Mathematics

1. Tim said that the binomial $x^2 + 16$ can be written as $x^2 - 16i^2$ and factored over the set of complex numbers. Do you agree with Tim? Explain why or why not.

2. Joshua said that the product of a complex number and its conjugate is always a real number. Do you agree with Joshua? Explain why or why not.

Developing Skills

In 3–17, find each sum or difference of the complex numbers in $a + bi$ form.

3. $(6 + 7i) + (1 + 2i)$ **4.** $(3 - 5i) + (2 + i)$ **5.** $(5 - 6i) + (4 + 2i)$

6. $(-3 + 3i) - (1 + 5i)$ **7.** $(2 - 8i) - (2 + 8i)$ **8.** $(4 + 12i) + (-4 - 2i)$

9. $(1 + 9i) - (1 + 2i)$ **10.** $(10 - 12i) - (12 + 7i)$ **11.** $(0 + 3i) + (0 - 3i)$

12. $\left(\frac{1}{2} + \frac{1}{2}i\right) + \left(\frac{1}{4} - \frac{3}{4}i\right)$ **13.** $\left(\frac{2}{3} - \frac{1}{6}i\right) + (2 - i)$ **14.** $(1 + 0i) - \left(\frac{1}{5} - \frac{2}{5}i\right)$

15. $\left(-\frac{3}{2} + \frac{5}{3}i\right) - \left(\frac{9}{4} - \frac{1}{5}i\right)$ **16.** $\left(\frac{1}{4} + 12i\right) + \left(7 + \frac{i}{10}\right)$ **17.** $\left(\frac{3}{7} + \frac{1}{3}i\right) + \left(\frac{1}{6} - \frac{7}{8}i\right)$

In 18–25, write the complex conjugate of each number.

18. $3 + 4i$ **19.** $2 - 5i$ **20.** $-8 + i$ **21.** $-6 - 9i$

22. $\frac{1}{2} - 3i$ **23.** $-4 + \frac{1}{3}i$ **24.** $\frac{5}{3} - \frac{2}{3}i$ **25.** $\pi + 2i$

In 26–37, find each product.

26. $(1 + 5i)(1 + 2i)$ **27.** $(3 + 2i)(3 + 3i)$ **28.** $(2 - 3i)^2$

29. $(4 - 5i)(3 - 2i)$ **30.** $(7 + i)(2 + 3i)$ **31.** $(-2 - i)(1 + 2i)$

32. $(-4 - i)(-4 + i)$ **33.** $(-12 - 2i)(12 - 2i)$ **34.** $(5 - 3i)(5 + 3i)$

35. $(3 - i)\left(\frac{3}{10} + \frac{1}{10}i\right)$ **36.** $\left(\frac{1}{5} - \frac{1}{10}i\right)(4 + 2i)$ **37.** $\left(-\frac{1}{8} + \frac{1}{8}i\right)(4 + 4i)$

In 38–45, find the multiplicative inverse of each of the following in $a + bi$ form.

38. $1 + i$ **39.** $2 + 4i$ **40.** $-1 + 2i$ **41.** $3 - 3i$

42. $\frac{1}{2} - \frac{1}{4}i$ **43.** $2 - \frac{1}{2}i$ **44.** $\frac{5}{6} + 3i$ **45.** $9 + \pi i$

In 46–60, write each quotient in $a + bi$ form.

46. $(8 + 4i) \div (1 + i)$ **47.** $(2 + 4i) \div (1 - i)$ **48.** $(10 + 5i) \div (1 + 2i)$

49. $(5 - 15i) \div (3 - i)$ **50.** $\frac{2 + 2i}{4 - 2i}$ **51.** $\frac{10 - 5i}{2 + 6i}$

52. $\frac{8 + 2i}{1 + 3i}$ **53.** $\frac{3 + i}{3 - i}$ **54.** $\frac{5 - 2i}{5 + 2i}$

55. $\frac{12 + 3i}{3i}$ **56.** $\frac{8 + 6i}{2i}$ **57.** $\frac{\frac{1}{5} - \frac{1}{5}i}{3 - 4i}$

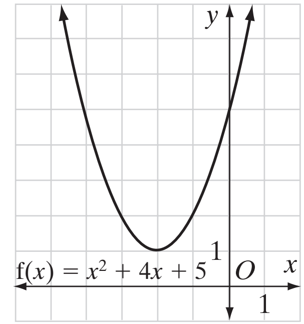
58. $\frac{1 + i}{\frac{1}{2} - \frac{1}{2}i}$ **59.** $\frac{3i}{\pi + \frac{1}{2}i}$ **60.** $\frac{\frac{1}{7} + i}{\frac{5}{i}}$

Applying Skills

61. Impedance is the resistance to the flow of current in an electric circuit measured in ohms. The impedance, Z, in a circuit is found by using the formula $Z = \frac{V}{I}$ where V is the voltage (measured in volts) and I is the current (measured in amperes). Find the impedance when $V = 1.8 - 0.4i$ volts and $I = -0.3i$ amperes.

62. Find the current that will flow when $V = 1.6 - 0.3i$ volts and $Z = 1.5 + 8i$ ohms using the formula $Z = \frac{V}{I}$.

5-6 COMPLEX ROOTS OF A QUADRATIC EQUATION



When we used the quadratic formula to find the roots of a quadratic equation, we found that some quadratic equations have no real roots. For example, the quadratic equation $x^2 + 4x + 5 = 0$ has no real roots since the graph of the corresponding quadratic function $f(x) = x^2 + 4x + 5$, shown on the right, has no x-intercepts. Can the roots of a quadratic equation be complex numbers?

We will use the quadratic formula to find the roots of the equation $x^2 + 4x + 5 = 0$. For this equation, $a = 1$, $b = 4$, and $c = 5$. The value of the discriminant is

$$b^2 - 4ac = 4^2 - 4(1)(5) \quad \text{or} \quad -4$$

Therefore, the roots are not real numbers.

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm 2i}{2} \\ &= -2 \pm i \end{aligned}$$

The roots are the complex numbers $-2 + i$ and $-2 - i$.

Therefore, every quadratic equation with rational coefficients has two complex roots. Those roots may be two real numbers. Complex roots that are real may be rational or irrational and equal or unequal. Roots that are not real numbers are complex conjugates.

Note that complex roots that are not real are often referred to as *imaginary roots* and complex roots of the form $0 + bi = bi$ as *pure imaginary roots*. A quadratic equation with rational coefficients has roots that are pure imaginary only if $b = 0$ and $b^2 - 4ac < 0$.

Let $ax^2 + bx + c = 0$ be a quadratic equation and a, b, and c be rational numbers with $a \neq 0$:

When the discriminant $b^2 - 4ac$ is	**The roots of the equation are**	**The number of x-intercepts of the function is:**
> 0 and a perfect square	real, rational, and unequal	2
> 0 and not a perfect square	real, irrational, and unequal	2
$= 0$	real, rational, and equal	1
< 0	imaginary numbers	0

EXAMPLE 1

Find the complex roots of the equation $x^2 + 12 = 6x$.

Solution *How to Proceed*

(1) Write the equation in standard form:
$$x^2 + 12 = 6x$$
$$x^2 - 6x + 12 = 0$$

(2) Determine the values of a, b, and c:
$$a = 1, b = -6, c = 12$$

(3) Substitute the values of a, b, and c into the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$$

(4) Perform the computation:
$$= \frac{6 \pm \sqrt{36 - 48}}{2}$$
$$= \frac{6 \pm \sqrt{-12}}{2}$$

(5) Write $\sqrt{-12}$ in simplest form in terms of i:
$$= \frac{6 \pm \sqrt{4}\sqrt{3}\sqrt{-1}}{2}$$
$$= \frac{6 \pm 2i\sqrt{3}}{2}$$
$$= 3 \pm i\sqrt{3}$$

Answer The roots are $3 + i\sqrt{3}$ and $3 - i\sqrt{3}$.

EXAMPLE 2

For what values of c does the equation $2x^2 + 3x + c = 0$ have imaginary roots?

Solution *How to Proceed*

(1) The equation has imaginary roots when $b^2 - 4ac < 0$. Substitute $a = 2$, $b = 3$:
$$b^2 - 4ac < 0$$
$$3^2 - 4(2)c < 0$$

(2) Isolate the term in c:
$$9 - 8c < 0$$
$$-8c < -9$$

(3) Solve the inequality. Dividing by a negative number reverses the order of the inequality:
$$\frac{-8c}{-8} > \frac{-9}{-8}$$
$$c > \frac{9}{8}$$

Answer The roots are imaginary when $c > \frac{9}{8}$.

Exercises

Writing About Mathematics

1. Emily said that when a and c are real numbers with the same sign and $b = 0$, the roots of the equation $ax^2 + bx + c = 0$ are pure imaginary. Do you agree with Emily? Justify your answer.

2. Noah said that if a, b, and c are rational numbers and $b^2 - 4ac < 0$, then the roots of the equation $ax^2 + bx + c = 0$ are complex conjugates. Do you agree with Noah? Justify your answer.

Developing Skills

In 3–14, use the quadratic formula to find the imaginary roots of each equation.

3. $x^2 - 4x + 8 = 0$
4. $x^2 + 6x + 10 = 0$
5. $x^2 - 4x + 13 = 0$
6. $2x^2 + 2x + 1 = 0$
7. $x^2 + 10x + 29 = 0$
8. $x^2 + 8x + 17 = 0$
9. $x^2 - 2x + 10 = 0$
10. $4x^2 - 4x + 5 = 0$
11. $4x^2 + 4x + 17 = 0$
12. $x^2 + 5 = 4x$
13. $4x - 7 = x^2$
14. $2x = x^2 + 3$

5-7 SUM AND PRODUCT OF THE ROOTS OF A QUADRATIC EQUATION

Writing a Quadratic Equation Given the Roots of the Equation

We know that when a quadratic equation is in standard form, the polynomial can often be factored in order to find the roots. For example, to find the roots of $2x^2 - 5x + 3 = 0$, we can factor the polynomial.

$$2x^2 - 5x + 3 = 0$$
$$(2x - 3)(x - 1) = 0$$

$$\begin{array}{r|r} 2x - 3 = 0 & x - 1 = 0 \\ 2x = 3 & x = 1 \\ x = \frac{3}{2} & \end{array}$$

We can work backward on each of these steps to write a quadratic equation with a given pair of factors. For example, write a quadratic equation whose roots are 3 and −5:

(1) Let x equal each of the roots: $x = 3 \mid x = -5$

(2) Write equivalent equations with one side equal to 0: $x - 3 = 0 \mid x + 5 = 0$

(3) If equals are multiplied by equals their products are equal:

$$(x - 3)(x + 5) = 0(0)$$
$$x^2 + 2x - 15 = 0$$

We can check to show that 3 and −5 are the roots of $x^2 + 2x - 15 = 0$:

Check for $x = 3$:

$$\begin{aligned} x^2 + 2x - 15 &= 0 \\ (3)^2 + 2(3) - 15 &\stackrel{?}{=} 0 \\ 9 + 6 - 15 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

Check for $x = -5$:

$$\begin{aligned} x^2 + 2x - 15 &= 0 \\ (-5)^2 + 2(-5) - 15 &\stackrel{?}{=} 0 \\ 25 - 10 - 15 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

We can use the same procedure for the general case. Write the quadratic equation that has roots r_1 and r_2:

(1) Let x equal each of the roots: $x = r_1 \mid x = r_2$

(2) Write equivalent equations with one side equal to 0: $x - r_1 = 0 \mid x - r_2 = 0$

(3) If equals are multiplied by equals, their products are equal:

$$(x - r_2)(x - r_1) = 0(0)$$
$$x^2 - r_1x - r_2x + r_1r_2 = 0$$

$$\mathbf{x^2 - (r_1 + r_2)x + r_1r_2 = 0}$$

In the example given above, $r_1 = 3$ and $r_2 = -5$. The equation with these roots is:

$$\begin{aligned} x^2 - (r_1 + r_2)x + r_1r_2 &= 0 \\ x^2 - [3 + (-5)]x + 3(-5) &= 0 \\ x^2 + 2x - 15 &= 0 \end{aligned}$$

EXAMPLE 1

Write an equation with roots $\frac{1 + \sqrt{6}}{3}$ and $\frac{1 - \sqrt{6}}{3}$.

Solution $r_1 + r_2 = \frac{1 + \sqrt{6}}{3} + \frac{1 - \sqrt{6}}{3} = \frac{2}{3}$

$$r_1r_2 = \left(\frac{1 + \sqrt{6}}{3}\right)\left(\frac{1 - \sqrt{6}}{3}\right) = \frac{1^2 - (\sqrt{6})^2}{9} = \frac{1 - 6}{9} = \frac{-5}{9} = -\frac{5}{9}$$

Therefore, the equation is $x^2 - \frac{2}{3}x - \frac{5}{9} = 0$ or $9x^2 - 6x - 5 = 0$.

Alternative Solution

(1) Let x equal each of the roots:

$$x = \left(\frac{1 + \sqrt{6}}{3}\right) \qquad x = \left(\frac{1 - \sqrt{6}}{3}\right)$$

(2) Write equivalent equations with one side equal to 0:

$$x - \left(\frac{1 + \sqrt{6}}{3}\right) = 0 \qquad x - \left(\frac{1 - \sqrt{6}}{3}\right) = 0$$

(3) If equals are multiplied by equals, their products are equal:

$$\left[x - \left(\frac{1 + \sqrt{6}}{3}\right)\right] \cdot \left[x - \left(\frac{1 - \sqrt{6}}{3}\right)\right] = 0$$

$$x^2 - \left(\frac{1 - \sqrt{6}}{3}\right)x - \left(\frac{1 + \sqrt{6}}{3}\right)x + \left(\frac{1 + \sqrt{6}}{3}\right)\left(\frac{1 - \sqrt{6}}{3}\right) = 0$$

$$x^2 - \tfrac{2}{3}x + \frac{1 - 6}{9} = 0$$

(4) Simplify:

$$x^2 - \tfrac{2}{3}x - \tfrac{5}{9} = 0$$

$$\text{or} \quad 9x^2 - 6x - 5 = 0$$

Answer $9x^2 - 6x - 5 = 0$

The Coefficients of a Quadratic Equation and Its Roots

Compare the equation $x^2 - (r_1 + r_2)x + r_1r_2 = 0$ derived previously with the standard form of the quadratic equation, $ax^2 + bx + c = 0$.

(1) Divide each side of the equation by a to write the general equation as an equivalent equation with the coefficient of x^2 equal to 1.

$$\tfrac{a}{a}x^2 + \tfrac{b}{a}x + \tfrac{c}{a} = \tfrac{0}{a}$$

$$x^2 + \tfrac{b}{a}x + \tfrac{c}{a} = 0$$

(2) Compare the equation in step 1 with $x^2 - (r_1 + r_2)x + r_1r_2 = 0$ by equating corresponding coefficients.

$$\tfrac{b}{a} = -(r_1 + r_2) \quad \text{or} \quad -\tfrac{b}{a} = r_1 + r_2$$

$$\tfrac{c}{a} = r_1r_2$$

We can conclude the following:

► **If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, then:**

$-\frac{b}{a}$ is equal to the sum of the roots $r_1 + r_2$.

$\frac{c}{a}$ is equal to the product of the roots r_1r_2.

Thus, we can find the sum and product of the roots of a quadratic equation without actually solving the equation.

Note that an equivalent equation of any quadratic equation can be written with the coefficient of x^2 equal to 1. For example, $3x^2 - 5x + 2 = 0$ and $x^2 - \frac{5}{3}x + \frac{2}{3} = 0$ are equivalent equations.

- The sum of the roots of $3x^2 - 5x + 2 = 0$ is $-\left(-\frac{5}{3}\right)$ or $\frac{5}{3}$.
- The product of the roots of $3x^2 - 5x + 2 = 0$ is $\frac{2}{3}$.

EXAMPLE 2

If one root of the equation $x^2 - 3x + c = 0$ is 5, what is the other root?

Solution The sum of the roots of the equation is $-\frac{b}{a} = -\frac{-3}{1} = 3$.

Let $r_1 = 5$. Then $r_1 + r_2 = 3$ or $r_2 = -2$.

Answer The other root is -2.

EXAMPLE 3

Find two numbers whose sum is 4 and whose product is 5.

Solution Let the numbers be the roots of the equation $ax^2 + bx + c = 0$.

The sum of the roots is $-\frac{b}{a} = 4$ and the product of the roots is $\frac{c}{a} = 5$.

We can choose any convenient value of a. Let $a = 1$.

Then:

$$-\frac{b}{1} = 4 \quad \text{or} \quad b = -4$$
$$\frac{c}{1} = 5 \quad \text{or} \quad c = 5$$

The equation is $x^2 - 4x + 5 = 0$.

Use the quadratic equation to find the roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

The numbers are $(2 + i)$ and $(2 - i)$.

Check The sum $(2 + i) + (2 - i) = 4$. ✔

The product $(2 + i)(2 - i) = 2^2 - i^2 = 4 - (-1) = 4 + 1 = 5$. ✔

Answer $(2 + i)$ and $(2 - i)$

Exercises

Writing About Mathematics

1. The roots of a quadratic equation with rational coefficients are $p \pm \sqrt{q}$. Write the equation in standard form in terms of p and q.

2. Adrien said that if the roots of a quadratic equation are $\frac{1}{2}$ and $\frac{3}{4}$, the equation is $4x^2 - 5x + \frac{3}{2} = 0$. Olivia said that the equation is $8x^2 - 10x + 3 = 0$. Who is correct? Justify your answer.

Developing Skills

In 3–17, without solving each equation, find the sum and product of the roots.

3. $x^2 + x + 1 = 0$ **4.** $x^2 + 4x + 5 = 0$ **5.** $2x^2 - 3x - 2 = 0$

6. $5x^2 + 2x - 10 = 0$ **7.** $3x^2 - 6x + 4 = 0$ **8.** $-x^2 + 3x + 1 = 0$

9. $8x + 12 = x^2$ **10.** $4x^2 = 2x + 9$ **11.** $2x^2 - 8 = 5x$

12. $x^2 - \frac{1}{4} = 0$ **13.** $x^2 + 1 = 0$ **14.** $x^2 + 2x = 0$

15. $8x^2 - 9 = -6x$ **16.** $x^2 + 2x - 3 = 0$ **17.** $\frac{3x^2 + 3x + 5}{3} = 0$

In 18–27, one of the roots is given. Find the other root.

18. $x^2 + 15x + c = 0; -5$ **19.** $x^2 - 8x + c = 0; -3$

20. $2x^2 + 3x + c = 0; 1$ **21.** $-4x^2 - 5x + c = 0; 2$

22. $-6x^2 + 2x + c = 0; \frac{1}{2}$ **23.** $6x^2 - x + c = 0; -\frac{2}{3}$

24. $m^2 - 4m + n = 0; 3$ **25.** $z^2 + 2z + k = 0; \frac{3}{4}$

26. $x^2 + bx + 3 = 0; 1$ **27.** $-7w^2 + bw - 5 = 0; -1$

28. One root of the equation $-3x^2 + 9x + c = 0$ is $\sqrt{2}$.
- **a.** Find the other root.
- **b.** Find the value of c.
- **c.** Explain why the roots of this equation are not conjugates.

29. One root of the equation $-x^2 - 11x + c = 0$ is $\sqrt{3}$.
- **a.** Find the other root.
- **b.** Find the value of c.
- **c.** Explain why the roots of this equation are not conjugates.

In 30–43, write a quadratic equation with integer coefficients for each pair of roots.

30. 2, 5 **31.** 4, 7 **32.** –3, 4 **33.** –2, −1

34. −3, 3 **35.** $\frac{1}{2}, \frac{7}{2}$ **36.** $\frac{3}{4}, -\frac{3}{8}$ **37.** 1, 0

38. $2 + \sqrt{3}, 2 - \sqrt{3}$ **39.** $\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}$ **40.** $\frac{-1 + \sqrt{3}}{3}, \frac{-1 - \sqrt{3}}{3}$

41. $3 + i, 3 - i$ **42.** $\frac{3 - 2i}{2}, \frac{3 + 2i}{2}$ **43.** $\frac{3}{2}i, -\frac{3}{2}i$

Applying Skills

44. One root of a quadratic equation is three more than the other. The sum of the roots is 15. Write the equation.

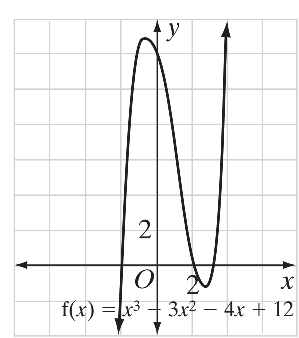
45. The difference between the roots of a quadratic equation is $4i$. The sum of the roots is 12. Write the equation.

46. Write a quadratic equation for which the sum of the roots is equal to the product of the roots.

47. Use the quadratic formula to prove that the sum of the roots of the equation $ax^2 + bx + c = 0$ is $-\frac{b}{a}$ and the product is $\frac{c}{a}$.

48. A root of $x^2 + bx + c = 0$ is an integer and b and c are integers. Explain why the root must be a factor of c.

5-8 SOLVING HIGHER DEGREE POLYNOMIAL EQUATIONS



The graph of the polynomial function $f(x) = x^3 - 3x^2 - 4x + 12$ is shown at the right. We know that the real roots of a polynomial function are the x-coordinates of the points at which the graph intersects the x-axis, that is, the values of x for which $y = f(x) = 0$. A polynomial function of degree three can have no more than three real roots. The graph appears to intersect the x-axis at −2, 2, and 3, but are these exact values of the roots? We know that we can find the rational roots of a polynomial function by factoring. We can also show that these are exact roots by substituting the roots in the equation to show that they satisfy the equation.

Solve by factoring:

Find a common binomial factor by first factoring the first two terms and then factoring the last two terms of the polynomial. Then factor out the common binomial factor.

$$\begin{aligned} f(x) &= x^3 - 3x^2 - 4x + 12 \\ 0 &= x^2(x - 3) - 4(x - 3) \\ 0 &= (x - 3)(x^2 - 4) \end{aligned}$$

Now factor the difference of two squares.

$$\begin{aligned} 0 &= (x - 3)(x^2 - 4) \\ 0 &= (x - 3)(x - 2)(x + 2) \end{aligned}$$

$x - 3 = 0$	$x - 2 = 0$	$x + 2 = 0$
$x = 3$	$x = 2$	$x = -2$

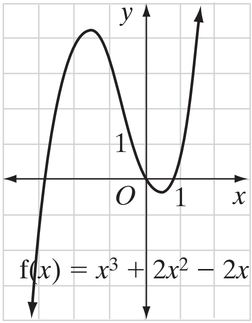
Solve by substitution:

$$\begin{aligned} x = 3: f(x) &= x^3 - 3x^2 - 4x + 12 \\ 0 &= 3^3 - 3(3)^2 - 4(3) + 12 \\ 0 &= 27 - 27 - 12 + 12 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x = 2: f(x) &= x^3 - 3x^2 - 4x + 12 \\ 0 &= 2^3 - 3(2)^2 - 4(2) + 12 \\ 0 &= 8 - 12 - 8 + 12 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x = -2: f(x) &= x^3 - 3x^2 - 4x + 12 \\ 0 &= (-2)^3 - 3(-2)^2 - 4(-2) + 12 \\ 0 &= -8 - 12 + 8 + 12 \\ 0 &= 0 \end{aligned}$$

Not every polynomial function has rational roots. The figure at the right shows the graph of the polynomial function $f(x) = x^3 + 2x^2 - 2x$. From the graph we see that this function appears to have a root between -3 and -2, a root at 0, and a root between 0 and 1. Can we verify that 0 is a root and can we find the exact values of the other two roots?



(1) Let $f(x) = 0$. Factor the polynomial if possible. We can factor out the common factor x:

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 2x \\ 0 &= x^3 + 2x^2 - 2x \\ 0 &= x(x^2 + 2x - 2) \end{aligned}$$

(2) The factor $x^2 + 2x - 2$ cannot be factored in the set of integers. Set each factor equal to 0. Use the quadratic formula to solve the quadratic equation:

(3) The roots of $f(x) = x^3 + 2x^2 - 2x$ are 0, $-1 + \sqrt{3}$, and $-1 - \sqrt{3}$:

$x = 0$

$$\begin{aligned} x^2 + 2x - 2 &= 0 \\ x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{12}}{2} \\ &= \frac{-2 \pm 2\sqrt{3}}{2} \\ &= -1 \pm \sqrt{3} \end{aligned}$$

We can use a calculator to approximate the values of $-1 + \sqrt{3}$ and $-1 - \sqrt{3}$:

$$-1 - \sqrt{3} \approx -2.7 \text{ and } -1 + \sqrt{3} \approx 0.7$$

These are values between -3 and -2 and between 0 and 1 as seen on the graph on the previous page. The function has three real roots. One of the roots, 0, is rational and the other two roots, $-1 + \sqrt{3}$ and $-1 - \sqrt{3}$, are irrational roots.

Note that in factoring a polynomial of degree three or greater, we look for the following factors:

1. a common monomial factor.
2. the binomial factors of a trinomial.
3. a common binomial factor.
4. factors of the difference of two squares.

EXAMPLE 1

Find three roots of the function $f(x) = x^3 - 2x^2 + 9x - 18$.

Solution *How to Proceed*

(1) Try to factor the polynomial. There is no common monomial factor. Look for a common binomial factor:

$$\begin{aligned} &x^3 - 2x^2 + 9x - 18 \\ &= x^2(x - 2) + 9(x - 2) \\ &= (x - 2)(x^2 + 9) \end{aligned}$$

(2) Let $f(x) = 0$:

$$f(x) = x^3 - 2x^2 + 9x - 18$$

(3) Set each factor equal to 0. and solve each equation for x:

$$0 = (x - 2)(x^2 + 9)$$

$$\begin{array}{r|r} x - 2 = 0 & x^2 + 9 = 0 \\ x = 2 & x^2 = -9 \\ & x = \pm\sqrt{-9} \\ & x = \pm\sqrt{9}\sqrt{-1} \\ & x = \pm 3i \end{array}$$

Answer The roots are 2, $3i$, and $-3i$.

EXAMPLE 2

Is 2 is a root of the function $f(x) = x^4 - 5x^2 + x + 2$?

Solution If 2 is a root of $f(x) = x^4 - 5x^2 + x + 2$, then $f(2) = 0$.

$$f(2) = 2^4 - 5(2)^2 + 2 + 2 = 16 - 20 + 2 + 2 = 0 \checkmark$$

Answer 2 is a root of $f(x) = x^4 - 5x^2 + x + 2$.

Exercises

Writing About Mathematics

1. Sharon said that if $f(x)$ is a polynomial function and $f(a) = 0$, then a is a root of the function. Do you agree with Sharon? Explain why or why not.

2. Jordan said that if the roots of a polynomial function $f(x)$ are r_1, r_2, and r_3, then the roots of $g(x) = f(x - a)$ are $r_1 + a$, $r_2 + a$, and $r_3 + a$. Do you agree with Jordan? Explain why or why not.

Developing Skills

In 3–18, find all roots of each given function by factoring or by using the quadratic formula.

3. $f(x) = x^3 + 7x^2 + 10x$
4. $f(x) = 2x^3 + 2x^2 - 4x$
5. $f(x) = x^3 + 3x^2 + 4x + 12$
6. $f(x) = x^3 - x^2 + 3x - 3$
7. $f(x) = 2x^3 - 3x^2 - 2x + 3$
8. $f(x) = -2x^3 + 6x^2 - x + 3$
9. $f(x) = x^4 - 5x^2 + 4$
10. $f(x) = x^4 + 5x^2 + 4$
11. $f(x) = x^4 - 81$
12. $f(x) = 16x^4 - 1$
13. $f(x) = x^4 - 10x^2 + 9$
14. $f(x) = x^5 - x^4 - 2x^3$
15. $f(x) = x^3 - 18x$
16. $f(x) = (2x^2 + x - 1)(x^2 - 3x + 4)$
17. $f(x) = (x^2 - 1)(3x^2 + 2x + 1)$
18. $f(x) = x^3 + 2x^2 - x - 2$

In 19–28: **a.** Find $f(a)$ for each given function. **b.** Is a a root of the function?

19. $f(x) = x^4 - 1$ and $a = 1$
20. $f(x) = x^3 + 4x$, and $a = -2$
21. $f(x) = 5x^2 + 4x + 1$ and $a = -1$
22. $f(x) = -x^3 + x - 24$ and $a = -3$
23. $f(x) = x^4 + x^2 + x + 1$ and $a = 0$
24. $f(x) = 2x^3 + 3x^2 - 1$ and $a = \frac{1}{2}$
25. $f(x) = x^3 - 3x^2 + x - 3$ and $a = i$
26. $f(x) = x^4 - 2x^2 + x$ and $a = \sqrt{3}$
27. $f(x) = x^3 - 2x + 3$ and $a = 2 + i$
28. $f(x) = -5x^3 + 5x^2 + 2x + 3$ and $a = \frac{3}{2}i$

Applying Skills

29. **a.** Verify by multiplication that $(x - 1)(x^2 + x + 1) = x^3 - 1$.

    b. Use the factors of $x^3 - 1$ to find the three roots of $f(x) = x^3 - 1$.

    c. If $x^3 - 1 = 0$, then $x^3 = 1$ and $x = \sqrt[3]{1}$. Use the answer to part **b** to write the three cube roots of 1. Explain your reasoning.

    d. Verify that each of the two imaginary roots of $f(x) = x^3 - 1$ is a cube root of 1.

30. a. Verify by multiplication that $(x + 1)(x^2 - x + 1) = x^3 + 1$.

b. Use the factors of $x^3 + 1$ to find the three roots of $f(x) = x^3 + 1$.

c. If $x^3 + 1 = 0$, then $x^3 = -1$ and $x = \sqrt[3]{-1}$. Use the answer to part **b** to write the three cube roots of -1. Explain your reasoning.

d. Verify that each of the two imaginary roots of $f(x) = x^3 + 1$ is a cube root of -1.

31. Let $f(x) = x^3 + 3x^2 - 2x - 6$ and $g(x) = 2f(x) = 2x^3 + 6x - 4x - 12$ be two cubic polynomial functions.

a. How does the graph of $f(x)$ compare with the graph of $g(x)$?

b. How do the roots of $f(x)$ compare with the roots of $g(x)$?

c. In general, if $p(x) = aq(x)$ and $a > 0$, how does the graph of $p(x)$ compare with the graph of $q(x)$?

d. How do the roots of $p(x)$ compare with the roots of $q(x)$?

Hands-On Activity

The following activity will allow you to evaluate a function for any constant and test possible roots of a function. This process is called **synthetic substitution**. Let $f(x) = x^4 - 3x^3 + x^2 - 2x + 3$. Find $f(2)$:

(1) List the coefficients of the terms of the function and the constant to be tested as shown:

$$\begin{array}{rrrrr|l} 1 & -3 & 1 & -2 & 3 & 2 \\ & & & & & \\ \hline & & & & & \end{array}$$

(2) Bring down the first coefficient as shown and then multiply it by the constant 2 and write the product under the second coefficient:

$$\begin{array}{rrrrr|l} 1 & -3 & 1 & -2 & 3 & 2 \\ & 2 & & & & \\ \hline 1 & & & & & \end{array}$$

(3) Add the second coefficient and the product found in step 2:

$$\begin{array}{rrrrr|l} 1 & -3 & 1 & -2 & 3 & 2 \\ & 2 & & & & \\ \hline 1 & -1 & & & & \end{array}$$

(4) Multiply this sum by the constant 2 and place the product under the next coefficient:

$$\begin{array}{rrrrr|l} 1 & -3 & 1 & -2 & 3 & 2 \\ & 2 & -2 & & & \\ \hline 1 & -1 & & & & \end{array}$$

(5) Add the coefficient and the product found in step 4:

$$\begin{array}{rrrrr|l} 1 & -3 & 1 & -2 & 3 & 2 \\ & 2 & -2 & & & \\ \hline 1 & -1 & -1 & & & \end{array}$$

(6) Repeat steps 4 and 5 until a final sum is found using the last coefficient:

$$\begin{array}{rrrrr|l} 1 & -3 & 1 & -2 & 3 & 2 \\ & 2 & -2 & -2 & -8 & \\ \hline 1 & -1 & -1 & -4 & -5 & \end{array}$$

The final number is -5. Therefore, $f(2) = -5$. Use a calculator to verify that this is true. Repeat the steps for $f(1)$. The final number should be 0. This means that $f(1) = 0$ and that 1 is a root of the function.

Repeat the process for the following function, using the numbers −3, −2, −1, 1, 2, and 3 for each function. Find the roots of the function.

a. $f(x) = x^3 - 2x^2 - x + 2$

b. $f(x) = x^3 - 3x^2 - 4x + 12$

c. $f(x) = x^3 - 7x - 6 = x^3 + 0x^2 - 7x - 6$ (Include 0 in the list of coefficients.)

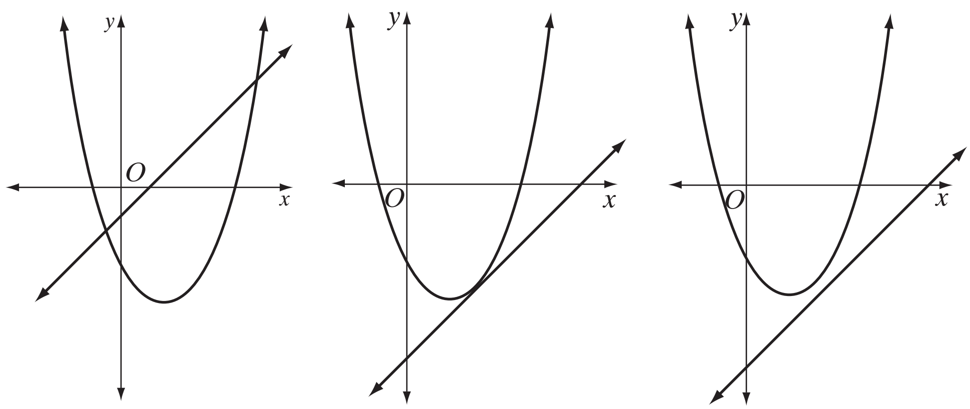
d. $f(x) = x^4 - 5x^2 + 4 = x^4 + 0x^3 - 5x^2 + 0x + 4$

5-9 SOLUTIONS OF SYSTEMS OF EQUATIONS AND INEQUALITIES

A system of equations is a set of two or more equations. The system is **consistent** if there exists at least one common solution in the set of real numbers. The solution set of a consistent system of equations can be found using a graphic method or an algebraic method.

Solving Quadratic-Linear Systems

A system that consists of a quadratic function whose graph is a parabola and a linear function whose graph is a straight line is a **quadratic-linear system** in two variables. In the set of real numbers, a quadratic-linear system may have two solutions, one solution, or no solutions. Each solution can be written as the coordinates of the points of intersection of the graph of the parabola and the graph of the line.



Two Solutions One Solution No Solution

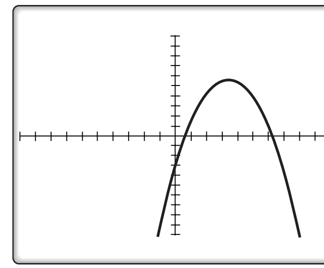
Recall that for the function $y = ax^2 + bx + c$, the x-coordinate of the turning point and the equation of the axis of symmetry is $x = \frac{-b}{2a}$. When a is positive, the parabola opens upward and has a minimum value of y as shown in the figures above. When a is negative, the parabola opens downward and has a maximum value of y as shown in the following example.

We can use either a graphic method or an algebraic method to find the common solutions of $y = -x^2 + 6x - 3$ and $y - x = 1$.

Graphic Method

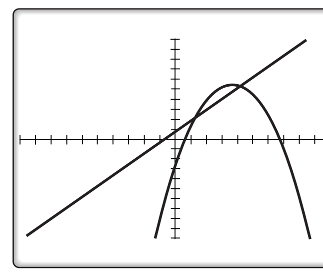
(1) Graph the parabola on the calculator. Choose a reasonable viewing window and enter $-x^2 + 6x - 3$ as Y_1.

ENTER: Y= CLEAR (-) X,T,Θ,n ^ 2 + 6 X,T,Θ,n − 3 GRAPH

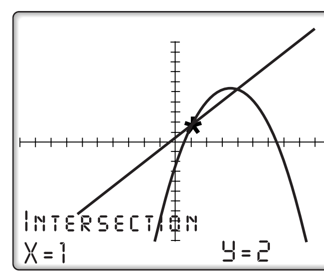


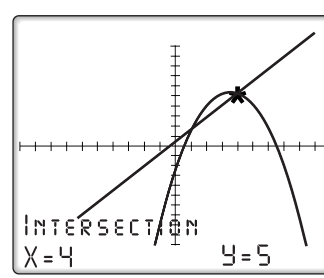
(2) On the same set of axes, draw the graph of $y - x = 1$. Write the equation in slope-intercept form: $y = x + 1$ and enter it as Y_2.

ENTER: Y= ▼ CLEAR X,T,Θ,n + 1 GRAPH



(3) The common solutions are the coordinates of the points of intersection of the graphs. We can find the intersection points by using the intersect function. Press 2nd CALC 5 ENTER ENTER to select both curves. When the calculator asks you for a guess, move the cursor near one of the intersection points using the arrow keys and then press ENTER. Repeat this process to find the other intersection point.





The common solutions are (1, 2) and (4, 5).

Note: The graphic method only gives exact values when the solutions are rational numbers. If the solutions are irrational, then the calculator will give approximations.

Algebraic Method

(1) Solve the linear equation for one of the variables in terms of the other. We will solve for y in terms of x:

$$y - x = 1$$
$$y = x + 1$$

(2) Replace y in the quadratic equation by its value in terms of x from the linear equation:

$$y = -x^2 + 6x - 3$$
$$x + 1 = -x^2 + 6x - 3$$

(3) Write the equation in standard form:

$$x^2 - 5x + 4 = 0$$

(4) Factor the trinomial:

$$(x - 4)(x - 1) = 0$$

(5) Set each factor equal to 0 and solve for x:

$x - 4 = 0$	$x - 1 = 0$
$x = 4$	$x = 1$

(6) Use the simplest equation in x and y to find the corresponding value of y for each value of x:

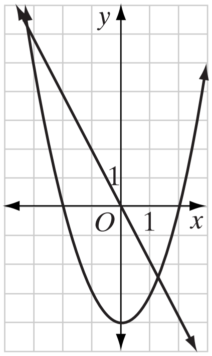
$y = x + 1$	$y = x + 1$
$y = 4 + 1$	$y = 1 + 1$
$y = 5$	$y = 2$

The common solutions are $x = 4, y = 5$ and $x = 1, y = 2$.

EXAMPLE 1

For the given system of equations, compare the solution that can be read from the graph with the algebraic solution.

$$y = x^2 - 4$$
$$y = -2x$$



Solution From the graph, the solutions appear to be $(-3.2, 6.5)$ and $(1.2, -2.5)$.

We can find exact values by using an algebraic solution.

(1) Substitute the value of y from the second equation in the first equation:

$$-2x = x^2 - 4$$

(2) Write the equation in standard form:

$$0 = x^2 + 2x - 4$$

(3) The polynomial $x^2 + 2x - 4$ cannot be factored in the set of integers. Use the quadratic formula with $a = 1, b = 2, c = -4$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4 + 16}}{2} \\ &= -1 \pm \sqrt{5} \end{aligned}$$

(4) Use the linear equation to find y for each value of x:

$y = -2x$	$y = -2x$
$y = -2(-1 - \sqrt{5})$	$y = -2(-1 + \sqrt{5})$
$y = 2 + 2\sqrt{5}$	$y = 2 - 2\sqrt{5}$

The exact solutions are $\left(-1 - \sqrt{5}, 2 + 2\sqrt{5}\right)$ and $\left(-1 + \sqrt{5}, 2 - 2\sqrt{5}\right)$. The rational approximations of these solutions, rounded to the nearest tenth, are the solutions that we found graphically.

EXAMPLE 2

Use an algebraic method to find the solution set of the equations:

$$y = 2x^2 - 4x - 5$$
$$3x - y = 1$$

Solution (1) Solve the linear equation for y:

$$\begin{aligned} 3x - y &= 1 \\ -y &= -3x + 1 \\ y &= 3x - 1 \end{aligned}$$

(2) Substitute for y in terms of x in the quadratic equation:

$$\begin{aligned} y &= 2x^2 - 4x - 5 \\ 3x - 1 &= 2x^2 - 4x - 5 \end{aligned}$$

(3) Write the equation in standard form and solve for x:

$$\begin{aligned} 0 &= 2x^2 - 7x - 4 \\ 0 &= (2x + 1)(x - 4) \end{aligned}$$

$$\begin{array}{l|l} 2x + 1 = 0 & x - 4 = 0 \\ 2x = -1 & x = 4 \\ x = -\frac{1}{2} & \end{array}$$

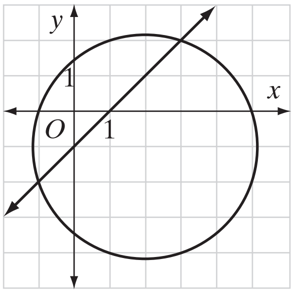
(4) Find the corresponding values of y using the linear equation solved for y:

$$\begin{array}{l|l} y = 3x - 1 & y = 3x - 1 \\ y = 3\left(-\frac{1}{2}\right) - 1 & y = 3(4) - 1 \\ y = -\frac{3}{2} - \frac{2}{2} & y = 12 - 1 \\ y = -\frac{5}{2} & y = 11 \end{array}$$

Answer $\left(-\frac{1}{2}, -\frac{5}{2}\right)$ and $(4, 11)$

EXAMPLE 3

The graph shows the circle whose equation is $(x - 2)^2 + (y + 1)^2 = 10$ and the line whose equation is $y = x - 1$.



a. Read the common solutions from the graph.

b. Check both solutions in both equations.

Solution **a.** The common solutions are $x = -1, y = -2$ and $x = 3, y = 2$.

b. *Check* $x = -1, y = -2$:

$$(x - 2)^2 + (y + 1)^2 = 10 \qquad y = x - 1$$
$$(-1 - 2)^2 + (-2 + 1)^2 \stackrel{?}{=} 10 \qquad -2 \stackrel{?}{=} -1 - 1$$
$$(-3)^2 + (-1)^2 \stackrel{?}{=} 10 \qquad -2 = -2 \checkmark$$
$$10 = 10 \checkmark$$

Check $x = 3, y = 2$:

$$(x - 2)^2 + (y + 1)^2 = 10 \qquad y = x - 1$$
$$(3 - 2)^2 + (2 + 1)^2 \stackrel{?}{=} 10 \qquad 2 = 3 - 1$$
$$(1)^2 + (3)^2 \stackrel{?}{=} 10 \qquad 2 = 2 \checkmark$$
$$10 = 10 \checkmark$$

Solving Quadratic Inequalities

You already know how to graph a linear inequality in two variables. The process for graphing a quadratic inequality in two variables is similar.

EXAMPLE 3

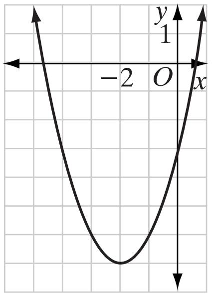
Graph the inequality $y + 3 \geq x^2 + 4x$.

Solution *How to Proceed*

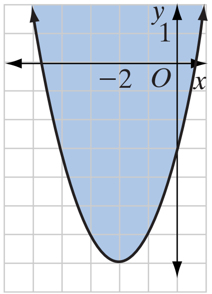
(1) Solve the inequality for y:

$$y + 3 \geq x^2 + 4x$$
$$y \geq x^2 + 4x - 3$$

(2) Graph the corresponding quadratic function $y = x^2 + 4x - 3$. Since the inequality is $\geq$, use a solid curve to indicate that the parabola is part of the solution:



(3) The values of $y > x^2 + 4x - 3$ are those in the region above the parabola. Shade this region:



(4) *Check:* Use any convenient test point to verify that the correct region has been shaded. Try (0, 0):

$$0 \stackrel{?}{\geq} 0^2 + 4(0) - 3$$
$$0 \geq -3 \checkmark$$

The test point satisfies the equation. The shaded region is the set of points whose coordinates make the inequality true.

In general:

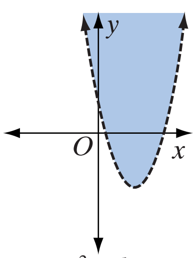
- **The solution set of the inequality $y > ax^2 + bx + c$ is the set of coordinates of the points above the graph of $y = ax^2 + bx + c$, which is drawn as a dotted line to show that it is not part of the solution set.**

- **The solution set of the inequality $y \geq ax^2 + bx + c$ is the set of coordinates of the points above the graph of $y = ax^2 + bx + c$, which is drawn as a solid line to show that it is part of the solution set.**

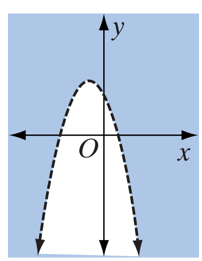
- **The solution set of the inequality $y < ax^2 + bx + c$ is the set of coordinates of the points below the graph of $y = ax^2 + bx + c$, which is drawn as a dotted line to show that it is not part of the solution set.**

- **The solution set of the inequality $y \leq ax^2 + bx + c$ is the set of coordinates of the points below the graph of $y = ax^2 + bx + c$, which is drawn as a solid line to show that it is part of the solution set.**

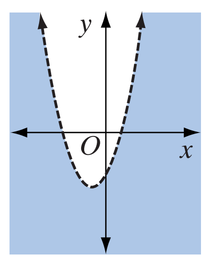
The following figures demonstrate some of these graphing rules:



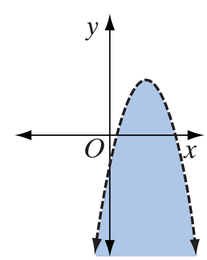
$y > ax^2 + bx + c$



$y > -ax^2 + bx + c$

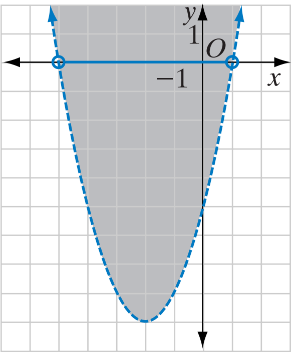


$y < ax^2 + bx + c$



$y < -ax^2 + bx + c$

Graphs can also be used to find the solution of a quadratic inequality in one variable. At the right is the graph of the equation $y = x^2 + 4x - 5$. The shaded region is the graph of $y > x^2 + 4x - 5$. Let $y = 0$. The graph of $y = 0$ is the x-axis. Therefore, the solutions of $0 > x^2 + 4x - 5$ are the x-coordinates of the points common to the graph of $y > x^2 + 4x - 5$ and the x-axis. From the graph on the right, we can see that the solution of $0 > x^2 + 4x - 5$ or $x^2 + 4x - 5 < 0$ is the interval $-5 < x < 1$.

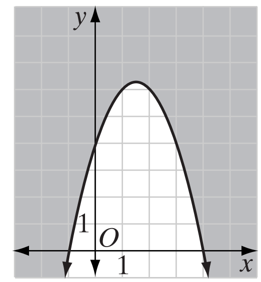


EXAMPLE 4

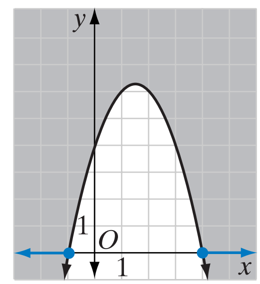
Solve the inequality $-x^2 + 3x + 4 \leq 0$.

Solution *How to Proceed*

(1) Let $y = 0$. The inequality $-x^2 + 3x + 4 \leq 0$ is equivalent to the intersection of $-x^2 + 3x + 4 \leq y$ and $y = 0$. Graph the quadratic function $y = -x^2 + 3x + 4$. Note that the graph intersects the x-axis at -1 and 4. Shade the area above the curve that is the graph of $y > -x^2 + 3x + 4$.



(2) The graph of $y = 0$ is the x-axis. The solution of $-x^2 + 3x + 4 \leq 0$ is the set of x-coordinates of the points common to graph of $y \geq -x^2 - 3x + 4$ and the x-axis.

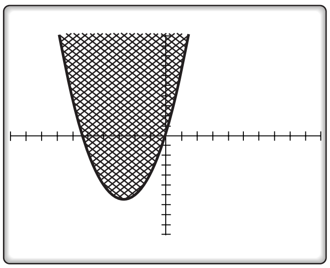


$x \leq -1$ or $x \geq 4$

Answer $x \leq -1$ or $x \geq 4$

Two Variable Inequalities and the Graphing Calculator

The graphing calculator can also be used to graph quadratic inequalities in two variables. For instance, to graph the inequality of Example 3, first rewrite the inequality with the x variable on the *right*: $y \geq x^2 + 4x - 3$. Enter the quadratic function $x^2 + 4x - 3$ into Y_1. Now move the cursor over the \ to the left of Y_1 using the arrow keys. Since the inequality is $\geq$, press **ENTER** until the cursor turns into ◥. Press **GRAPH** to graph the inequality.



Note: For $\geq$ and $>$ inequalities, use ◥. For $\leq$ and $<$ inequalities, use ◣.

Exercises

Writing About Mathematics

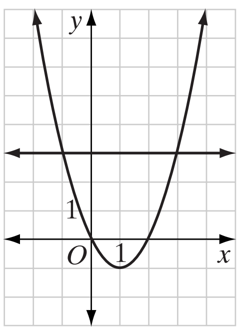
1. Explain the relationship between the solutions of $y > ax^2 + bx + c$ and the solutions of $0 > ax^2 + bx + c$.

2. Explain why the equations $y = x^2 + 2$ and $y = -2$ have no common solution in the set of real numbers.

Developing Skills

In 3–8, determine each common solution from the graph.

3.



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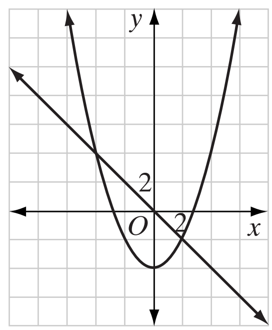

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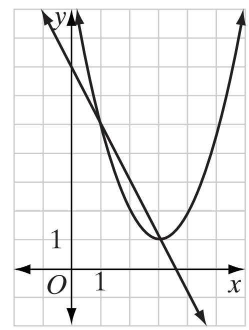

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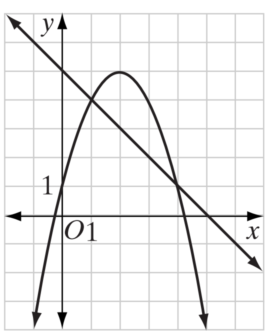

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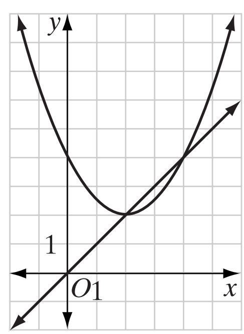

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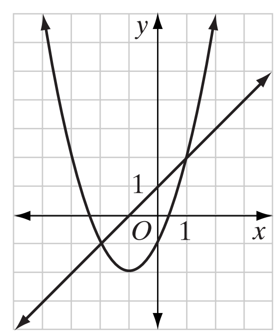


In 9–17, graph each system and determine the common solution from the graph.

9. $y = x^2 - 2x - 1$
$y = x + 3$

10. $y = x^2 + 2x$
$y = 2x + 1$

11. $-x^2 + 4x - y - 2 = 0$
$x + y = 4$

12. $y = -x^2 + 6x - 1$
$y = x + 3$

13. $y = 2x^2 + 2x + 3$
$y - x = 3$

14. $y = 2x^2 - 6x + 5$
$y = x + 2$

15. $x^2 - 4x - y + 4 = 0$
$y = \frac{4x + 7}{4}$

16. $\frac{x - 1}{y} = \frac{6}{x + 12}$
$y = x + 2$

17. $\frac{y}{x} = \frac{x + 7}{5}$
$y = 2x$

In 18–35, find each common solution algebraically. Express irrational roots in simplest radical form.

18. $y = x^2 - 2x$
$y = 3x$

19. $y = x^2 + 4x$
$2x - y = 1$

20. $y = x^2 + 2x + 3$
$x + y = 1$

21. $y = x^2 - 8x + 6$
$2x - y = 10$

22. $2x^2 - 3x + 3 - y = 0$
$y - 2x = 1$

23. $y = x^2 - 4x + 5$
$2y = x + 6$

24. $y = 2x^2 - 6x + 7$
$y = x + 4$

25. $x^2 + x - y = 7$
$\frac{1}{2}x = y + 2$

26. $y = 4x^2 - 6x - 10$
$y = 25 - 2x$

27. $y = x^2 - 2x + 1$
$y = -\frac{9x - 35}{2}$

28. $y = x^2 + 4x + 4$
$y = 4x + 6$

29. $2x^2 + x - y + 1 = 0$
$x - y + 7 = 0$

30. $y = x^2 + 2$
$2x - y = -3$

31. $y = 2x^2 + 3x + 4$
$y = 11x + 6$

32. $5x^2 + 3x - y = -1$
$10x + 1 = y$

33. $x^2 + y^2 = 24$
$x + y = 8$

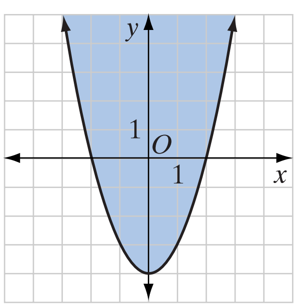
34. $x^2 + y^2 = 20$
$3x - y = 10$

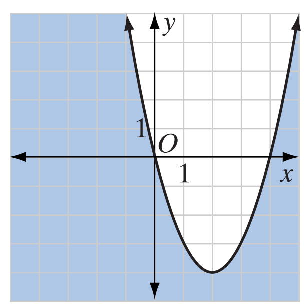
35. $x^2 + y^2 = 16$
$y = 2x$

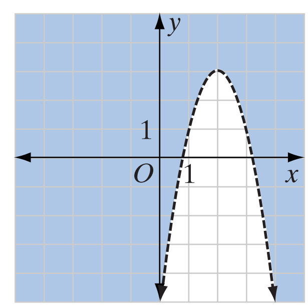
36. Write the equations of the graphs shown in Exercise 4 and solve the system algebraically. ($a = 1$)

37. Write the equations of the graphs shown in Exercise 6 and solve the system algebraically. ($a = -1$)

In 38–43, match the inequality with its graph. The graphs are labeled (1) to (6).

(1)


(2)


(3)


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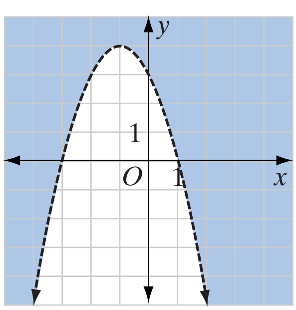

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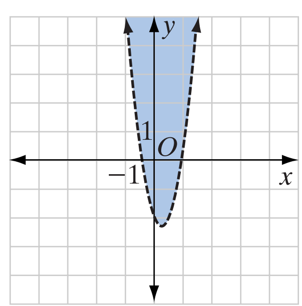
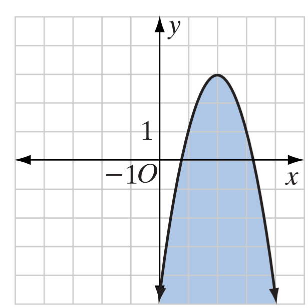


(6)


38. $x^2 \leq 4 + y$

39. $x^2 > -2x + 3 - y$

40. $y + 4x \leq x^2$

41. $y > -2(x - 2)^2 + 3$

42. $y \leq -2(x - 2)^2 + 3$

43. $4x^2 - 2x - y - 2 < 0$

In 44–51: **a.** Graph the given inequality. **b.** Determine if the given point is in the solution set.

44. $x^2 \leq 2x + y; (5, 4)$

45. $x^2 + 5 \geq y; (-1, 3)$

46. $x(x - 13) > y; (-5, 130)$

47. $x^2 - 18 \geq y + 3x; (0, -18)$

48. $2(x - 3)^2 + 3 < y; \left(\frac{1}{2}, 6\right)$

49. $-4(x + 2)^2 - 5 \leq y; \left(1, \frac{2}{3}\right)$

50. $4x^2 - 4x < 3 + y; \left(0, \frac{5}{3}\right)$

51. $6x^2 + x \leq 2 - y; (10, 0)$

Applying Skills

52. The area of a rectangular rug is 48 square feet. The length of the rug is 2 feet longer than the width. What are the dimensions of the rug?

53. The difference in the lengths of the sides of two squares is 1 meter. The difference in the areas of the squares is 13 square meters. What are the lengths of the sides of the squares?

54. The perimeter of a rectangle is 24 feet. The area of the rectangle is 32 square feet. Find the dimensions of the rectangle.

55. The endpoints of a diameter of a circle are (0, 0) and (8, 4).

a. Write an equation of the circle and draw its graph.

b. On the same set of axes, draw the graph of $x + y = 4$.

c. Find the common solutions of the circle and the line.

d. Check the solutions in both equations.

56. a. On the same set of axes, sketch the graphs of $y = x^2 - 4x + 5$ and $y = 2x + 2$.

b. Does the system of equations $y = x^2 - 4x + 5$ and $y = 2x + 2$ have a common solution in the set of real numbers? Justify your answer.

c. Find the solution set.

57. a. On the same set of axes, sketch the graphs of $y = x^2 + 5$ and $y = 2x$.

b. Does the system of equations $y = x^2 + 5$ and $y = 2x$ have a common solution in the set of real numbers? Justify your answer.

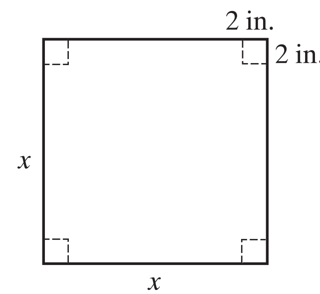
c. Does the system of equations $y = x^2 + 5$ and $y = 2x$ have a common solution in the set of complex numbers? If so, find the solution.

58. The graphs of the given equations have three points of intersection. Use an algebraic method to find the three solutions of this system of equations:

$$y = x^3 - 2x + 1$$
$$y = 2x + 1$$

59. A soccer ball is kicked upward from ground level with an initial velocity of 52 feet per second. The equation $h(t) = -16t^2 + 52t$ gives the ball's height in feet after t seconds. To the nearest tenth of a second, during what period of time was the height of the ball at least 20 feet?

60. A square piece of cardboard measuring x inches by x inches is to be used to form an open box by cutting off 2-inch squares, as shown in the figure.



a. Express the length of the sides of the base of the box in terms of x.

b. Write a function V(x) that represents the volume.

c. If the volume of the box must be greater than 128 cubic inches, what are the minimum dimensions of the square cardboard that can be used?

61. The profit function, in thousands of dollars, for a company that makes graphing calculators is $P(x) = -5x^2 + 5{,}400x - 106{,}000$ where x is the number of calculators sold in the millions.

a. Graph the profit function P(x).

b. How many calculators must the company sell in order to make a profit?

CHAPTER SUMMARY

The real **roots** of a polynomial function are the x-coordinates of the points where the graph of the function intersects the x-axis.

Any quadratic equation with rational roots can be solved by factoring over the set of integers.

Any quadratic equation can be solved by **completing the square**:

1. Write an equivalent equation with only the terms in x on one side.

2. Add the square of one-half the coefficient of x to both sides of the equation.

3. Take the square root of both sides of the equation.

4. Solve for x.

Any quadratic equation can be solved by using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When a, b, and c are rational numbers, the value of the **discriminant**, $b^2 - 4ac$, determines the nature of the roots and the number of x-intercepts of the quadratic function.

When the discriminant $b^2 - 4ac$ is:	The roots of the equation are:	The number of x-intercepts of the function is:
> 0 and a perfect square	real, rational, and unequal	2
> 0 and not a perfect square	real, irrational, and unequal	2
$= 0$	real, rational, and equal	1
< 0	imaginary numbers	0

If r_1 and r_2 are the roots of a quadratic equation, the equation is

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

For the quadratic equation $ax^2 + bx + c = 0$:

- the sum of the roots is $-\frac{b}{a}$.
- the product of the roots is $\frac{c}{a}$.

A number of the form $a\sqrt{-1} = ai$ is a **pure imaginary number** when a is a non-zero real number.

For any integer n:

$$i^{4n} = 1 \qquad i^{4n+1} = i \qquad i^{4n+2} = -1 \qquad i^{4n+3} = -i$$

A **complex number** is a number of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. A complex number that is not a real number is an imaginary number.

The identity element for addition is the real number $(0 + 0i)$ or 0.

The additive inverse of $(a + bi)$ is $(-a - bi)$.

The identity for multiplication is the real number $(1 + 0i)$ or 1.

The multiplicative inverse of $a + bi$ is $\frac{1}{a + bi}$ or $\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$.

The **complex conjugate** of $a + bi$ is $a - bi$.

A polynomial of degree three or greater can be factored using:

- a common monomial factor.
- the binomial factors of a trinomial.
- a common binomial factor.
- factors of the difference of two squares.

A system of equations is a set of two or more equations. The system is **consistent** if there exists at least one common solution in the set of real numbers. The solution set of a consistent system of equations can be found using a graphic method or an algebraic method.

The graph of a quadratic function can be used to estimate the solution of a quadratic inequality.

The solution set of the inequality $y > ax^2 + bx + c$ is the set of coordinates of the points above the graph of $y = ax^2 + bx + c$.

The solution set of the inequality $y < ax^2 + bx + c$ is the set of coordinates of the points below the graph of $y = ax^2 + bx + c$.

The solution set of $0 > ax^2 + bx + c$ is the set x-coordinates of points common to the graph of $y > ax^2 + bx + c$ and the x-axis.

VOCABULARY

5-1 Completing the square

5-2 Quadratic formula

5-3 Discriminant • Double root

5-4 i • Set of imaginary numbers • Pure imaginary number • Complex number

5-5 Complex conjugate

5-8 Synthetic substitution

5-9 Consistent • Quadratic-linear system

REVIEW EXERCISES

In 1–8, write each number in simplest form in terms of i.

1. $\sqrt{-1}$ **2.** $\sqrt{-16}$ **3.** $\sqrt{-9}$ **4.** $\sqrt{-12}$

5. $\sqrt{-4} + \sqrt{-25}$ **6.** $\sqrt{-18} + \sqrt{-32}$ **7.** $\sqrt{-64}\left(\sqrt{-\frac{1}{4}}\right)$ **8.** $\frac{\sqrt{-128}}{\sqrt{-12}}$

In 9–28, perform each indicated operation and express the result in $a + bi$ form.

9. $(2 + 3i) + (5 - 4i)$ **10.** $(1 + 2i) + (-1 + i)$

11. $(2 + 7i) + (2 - 7i)$ **12.** $(3 - 4i) + (-3 + 4i)$

13. $(1 + 2i) - (5 + 4i)$ **14.** $(8 - 6i) - (-2 - 2i)$

15. $(7 - 5i) - (7 + 5i)$ **16.** $(-2 + 3i) - (-2 - 3i)$

17. $(1 + 3i)(5 - 4i)$ **18.** $(2 + 6i)(3 - i)$

19. $(9 - i)(9 - i)$ **20.** $3i(4 - 2i)$

21. $\left(\frac{1}{2} - i\right)(2 + i)$ **22.** $\left(\frac{1}{5} - \frac{2}{5}i\right)(1 + 2i)$

23. $\frac{2 + 2i}{2i}$ **24.** $\frac{2 + 2i}{1 + i}$

25. $\frac{2 + 3i}{2 - 3i}$ **26.** $\frac{1 - i}{3 - i}$

27. $(1 + 4i)^2$ **28.** $(3 - 2i)^2$

In 29–36, find the roots of each equation by completing the square.

29. $x^2 - 7x + 1 = 0$ **30.** $x^2 - x - 12 = 0$

31. $x^2 + 4x + 5 = 0$ **32.** $x^2 - 6x - 10 = 0$

33. $x^2 - 6x + 10 = 0$ **34.** $\frac{x}{12} = \frac{5}{2x + 7}$

35. $3x^2 - 6x + 6 = 0$ **36.** $2x^2 + 3x - 2 = 0$

In 37–44, find the roots of each equation using the quadratic formula. Express irrational roots in simplest radical form.

37. $x^2 = x + 1$ **38.** $2x^2 - 2x = 1$

39. $5x^2 = 2x + 1$ **40.** $4x^2 - 12x + 13 = 0$

41. $x^3 - 2x^2 - 16x + 32 = 0$

42. $x^4 - 5x^2 + 4 = 0$

43. $0.1x^2 + 2x + 50 = 0$

44. $-3x^2 + \frac{1}{2}x + \frac{5}{3} = 0$

In 45–48, find the roots of the equation by any convenient method.

45. $x^3 - 6x^2 + 4x = 0$

46. $x^4 - 10x^2 + 9 = 0$

47. $3x^3 + 12x^2 - x - 4 = 0$

48. $(3x + 5)(x^2 + 5x - 6) = 0$

In 49–52, without graphing the parabola, describe the translation, reflection, and/or scaling that must be applied to $y = x^2$ to obtain the graph of each given function.

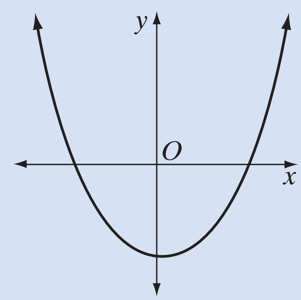
49. $y = x^2 + 2x + 2$

50. $y = x^2 + 3x + 10$

51. $y = 4x^2 - 6x + 3$

52. $y = -2x^2 + 5x - 10$

53. The graph on the right is the parabola $y = ax^2 + bx + c$ with a, b, and c rational numbers.



a. Describe the roots of the equation $0 = ax^2 + bx + c$ if the discriminant is 49.

b. Describe the roots of the equation $0 = ax^2 + bx + c$ if the discriminant is 32.

c. Can the discriminant be 0 or negative?

54. Let $h(x) = -20.5x^2 + 300.1x - 500$. Use the discriminant to determine if there exist real numbers, x, such that $h(x) = 325$.

In 55–60, find each common solution graphically.

55. $y = x^2 - 4$
$x + y = 2$

56. $y = -x^2 + 6x - 5$
$y = x - 1$

57. $y = 2x^2 - 8x$
$y = 2x$

58. $x^2 + y^2 = 25$
$x + y = 7$

59. $(x - 3)^2 + y^2 = 9$
$x + y = 6$

60. $(x + 1)^2 + (y - 2)^2 = 4$
$y = 3 - x$

In 61–70, find each common solution algebraically. Express irrational roots in simplest radical form.

61. $y = x^2 - 2x - 2$
$x + y = 4$

62. $y = x^2 - 8$
$2x - y = 0$

63. $x^2 + y = 3x$
$x - y = 3$

64. $\frac{x}{9} = \frac{8}{y}$
$y = 36 - 4x$

65. $x^2 = y + 5x$
$2x - y = 5$

66. $y = -x^2 + 2$
$x = y - 1$

67. $3x^2 - 2x - y = 0$
$3x + 5 = 2y$

68. $y = x^2 - 3x + 3$
$y = -x + 1$

69. $y = x^2 + 6x + 4$
$y = 2x - 1$

70. $x^2 + y^2 = 5$
$x - y - 1 = 0$

In 71–75, write each quadratic equation that has the given roots.

71. -3 and 5

72. $\frac{1}{2}$ and -4

73. $\sqrt{5}$ and $-\sqrt{5}$

74. $5 + 3\sqrt{2}$ and $5 - 3\sqrt{2}$

75. $6 + 2i$ and $6 - 2i$

76. For what value of c does the equation $x^2 - 6x + c = 0$ have equal roots?

77. For what values of b does $2x^2 + bx + 2 = 0$ have imaginary roots?

78. For what values of c does $x^2 - 3x + c = 0$ have real roots?

In 79 and 80: **a.** Graph the given inequality. **b.** Determine if the given point is in the solution set.

79. $y - (x + 2)^2 + 5 \geq 0$; $(2, 3)$

80. $-2x^2 + 3x < y$; $(0, \sqrt{5})$

81. The perimeter of a rectangle is 40 meters and the area is 97 square meters. Find the dimensions of the rectangle to the nearest tenth.

82. The manager of a theater is trying to determine the price to charge for tickets to a show. If the price is too low, there will not be enough money to cover expenses. If the price is too high, they may not get enough playgoers. The manager estimates that the profit, P, in hundreds of dollars per show, can be represented by $P = -(t - 12)^2 + 100$ where t is the price of a ticket in dollars. The manager is considering charging between \$0 and \$24.

a. Graph the profit function for the given range of prices.

b. The theater breaks even when profit is zero. For what ticket prices does the theater break even?

c. What price results in maximum profit? What is the maximum profit?

83. Pam wants to make a scarf that is 20 inches longer than it is wide. She wants the area of the scarf to be more than 300 square inches. Determine the possible dimensions of the scarf.

Exploration

1. Find the three binomial factors of $x^3 - x^2 - 4x + 4$ by factoring.

2. One of these factors is $(x - 2)$. Write $x^3 - x^2 - 4x + 4$ as the product of $(x - 2)$ and the trinomial that is the product of the other two factors.

3. Use synthetic substitution (see the Hands-On Activity on page 228) to show that 2 is a root of $f(x) = x^3 - x^2 - 4x + 4$.

4. Compare the coefficients of the trinomial from step 2 with the first three numbers of the synthetic substitution in step 3.

In (1)–(6), test each of the given polynomial functions to see if the relationship above appears to be true.

a. Use synthetic substitution to find a root r. (Try integers that are factors of the constant term.)

b. Write $(x - r)$ as one factor and use the first three numbers from the bottom line of the synthetic substitution as the coefficients of a trinomial factor.

c. Multiply $(x - r)$ times the trinomial factor written in step 2. Is the product equal to the given polynomial?

d. Write the three factors of the given polynomial and the three roots of the given function.

(1) $f(x) = x^3 - 6x^2 + 11x - 6$ (3) $f(x) = x^3 - 5x^2 + 7x - 3$

(2) $f(x) = x^3 + 2x^2 - 5x - 6$ (4) $f(x) = x^3 - 4x^2 + 5x - 2$

CUMULATIVE REVIEW — CHAPTERS 1–5

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. In simplest form, $3(1 - 2x)^2 - (x + 2)$ is equal to

(1) $36x^2 - 37x + 7$ (3) $12x^2 - 13x + 1$

(2) $36x^2 - 35x + 11$ (4) $12x^2 - 13x + 5$

2. The solution set of $x^2 - 2x - 8 = 0$ is

(1) $\{2, 4\}$ (3) $\{2, -4\}$

(2) $\{-2, 4\}$ (4) $\{-2, -4\}$

3. In simplest form, the fraction $\dfrac{1 - \frac{1}{2}}{2 + \frac{3}{4}}$ is equal to

(1) $\frac{1}{11}$ (2) $\frac{2}{11}$ (3) $\frac{3}{11}$ (4) $\frac{6}{11}$

4. The sum of $\left(2 - \sqrt{18}\right)$ and $\left(-4 + \sqrt{50}\right)$ is

(1) $-2 + 2\sqrt{2}$ (3) $-2 + 4\sqrt{2}$

(2) $-2 + \sqrt{32}$ (4) $-2 - 8\sqrt{2}$

5. Which of the following products is a rational number?

(1) $(2 + \sqrt{2})(2 + \sqrt{2})$ (3) $\sqrt{2}(2 + \sqrt{2})$

(2) $(2 + \sqrt{2})(2 - \sqrt{2})$ (4) $2(2 - \sqrt{2})$

6. If the graph of $g(x)$ is the graph of $f(x) = x^2$ moved 2 units to the right, then $g(x)$ is equal to

(1) $x^2 + 2$ (2) $x^2 - 2$ (3) $(x - 2)^2$ (4) $(x + 2)^2$

7. Which of the following is not a one-to-one function when the domain and range are the largest set of real numbers for which y is defined?

(1) $y = 2x + 3$ (2) $y = \frac{2}{x}$ (3) $y = |x|$ (4) $y = \sqrt{x}$

8. The equation of a circle with center at $(1, -1)$ and radius 2 is

(1) $(x - 1)^2 + (y + 1)^2 = 2$ (3) $(x + 1)^2 + (y - 1)^2 = 2$

(2) $(x - 1)^2 + (y + 1)^2 = 4$ (4) $(x + 1)^2 + (y - 1)^2 = 4$

9. The equation of the axis of symmetry of the graph of $y = 2x^2 - 4x + 7$ is

(1) $x = 1$ (2) $x = -1$ (3) $x = 2$ (4) $x = -2$

10. The graph of $f(x)$ is the graph of $f(x - 4)$ under the translation

(1) $T_{4,0}$ (2) $T_{-4,0}$ (3) $T_{0,4}$ (4) $T_{0,-4}$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Write the quotient $\frac{2 - i}{3 - i}$ in $a + bi$ form.

12. For what value(s) of x does $\frac{x - 3}{2} = \frac{2x + 1}{3}$?

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Write $\frac{3 + \sqrt{5}}{3 - \sqrt{5}}$ with a rational denominator.

14. What are the roots of the function $f(x) = 3 - |2x|$?

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. a. Sketch the graph of $y = -x^2 + 4x$.

b. Shade the region that is the solution set of $y \leq -x^2 + 4x$.

16. Let $f(x) = 2x + 4$ and $g(x) = x^2$.

a. Find $f \circ g(-3)$.

b. Find an expression for $h(x) = f \circ g(x)$.