CHAPTER

1

THE INTEGERS

In golf tournaments, a player's standing after each hole is often recorded on the leaderboard as the number of strokes above or below a standard for that hole called a *par*. A player's standing is a positive number if the number of strokes used was greater than par and a negative number if the number of strokes used was less than par. For example, if par for the first hole is 4 strokes and a player uses only 3, the player's standing after playing the first hole is -1.

Rosie Barbi is playing in an amateur tournament. Her standing is recorded as 2 below par (-2) after sixteen holes. She shoots 2 below par on the seventeenth hole and 1 above par on the eighteenth. What is Rosie's standing after eighteen holes? Nancy Taylor, who is her closest opponent, has a standing of 1 below par (-1) after sixteen holes, shoots 1 below par on the seventeenth hole and 1 below par on the eighteenth. What is Nancy's standing after eighteen holes?

In this chapter, we will review the set of integers and the way in which the integers are used in algebraic expressions, equations, and inequalities.

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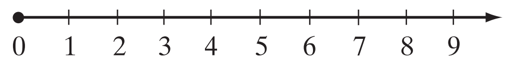
1-1 WHOLE NUMBERS, INTEGERS, AND THE NUMBER LINE

The first numbers that we learned as children and probably the first numbers used by humankind are the natural numbers. Most of us began our journey of discovery of the mathematical world by *counting*, the process that lists, in order, the names of the **natural numbers** or the **counting numbers**. When we combine the natural numbers with the number 0, we form the set of **whole numbers**:

$$\{0, 1, 2, 3, 4, 5, 6, \ldots\}$$

These numbers can be displayed as points on the number line:



The number line shows us the order of the whole numbers; 5 is to the right of 2 on the number line because $5 > 2$, and 3 is to the left of 8 on the number line because $3 < 8$. The number 0 is the smallest whole number. There is no largest whole number.

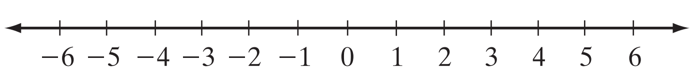
The temperature on a winter day may be two degrees above zero or two degrees below zero. The altitude of the highest point in North America is 20,320 feet above sea level and of the lowest point is 282 feet below sea level. We represent numbers less than zero by extending the number line to the left of zero, that is, to numbers that are less than zero, and by assigning to every whole number a an opposite, $-a$, such that $a + (-a) = 0$.

DEFINITION

The **opposite** or **additive inverse** of a is $-a$, the number such that

$$a + (-a) = 0.$$

The set of **integers** is the union of the set of whole numbers and their opposites. The set of non-zero whole numbers is the positive integers and the opposites of the positive integers are the negative integers.



Let a, b, and c represent elements of the set of integers. Under the operation of addition, the following properties are true:

1. Addition is closed: $a + b$ is an integer

2. Addition is commutative: $a + b = b + a$

3. Addition is associative: $(a + b) + c = a + (b + c)$

4. Addition has an identity element, 0: $a + 0 = a$

5. Every integer has an inverse: $a + (-a) = 0$

We say that the integers form a **commutative group** under addition because the five properties listed above are true for the set of integers.

Subtraction

DEFINITION

$a - b = c$ if and only if $b + c = a$.

Solve the equation $b + c = a$ for c:

$$\begin{aligned} b + c &= a \\ -b + b + c &= a + (-b) \\ c &= a + (-b) \end{aligned}$$

Therefore, $a - b = a + (-b)$.

Absolute Value

A number, a, and its opposite, $-a$, are the same distance from zero on the number line. When that distance is written as a positive number, it is called the **absolute value** of a.

- If $a > 0$, then $|a| = a - 0 = a$
- If $a < 0$, then $|a| = 0 - a = -a$

Note: When $a < 0$, a is a negative number and its opposite, $-a$, is a positive number.

For instance, $5 > 0$. Therefore, $|5| = 5 - 0 = 5$.

$-5 < 0$. Therefore, $|-5| = 0 - (-5) = 5$.

We can also say that $|a| = |-a| = a$ or $-a$, whichever is positive.

EXAMPLE 1

Show that the opposite of $-b$ is b.

Solution The opposite of b, $-b$, is the number such that $b + (-b) = 0$.

Since addition is commutative, $b + (-b) = (-b) + b = 0$.

The opposite of $-b$ is the number such that $(-b) + b = 0$. Therefore, the opposite of $-b$ is b.

Exercises

Writing About Mathematics

1. Tina is three years old and knows how to count. Explain how you would show Tina that $3 + 2 = 5$.

2. Greg said that $|a - b| = |b - a|$. Do you agree with Greg? Explain why or why not.

Developing Skills

In 3–14, find the value of each given expression.

3. $|6|$ **4.** $|-12|$ **5.** $|8 - 3|$

6. $|3 - 8|$ **7.** $|5 + (-12)|$ **8.** $|-12 + (-(-5))|$

9. $|4 - 6 + (-2)|$ **10.** $|8 + (10 - 18)|$ **11.** $|3| - |3|$

12. $|8| - |-2| - |2|$ **13.** $-(|-2| + |3|)$ **14.** $|4 - 3| + |-1|$

In 15–18, use the definition of subtraction to write each subtraction as a sum.

15. $8 - 5 = 3$ **16.** $7 - (-2) = 9$

17. $-2 - 5 = -7$ **18.** $-8 - (-5) = -3$

19. Two distinct points on the number line represent the numbers a and b. If $|5 - a| = |5 - b| = 6$, what are the values of a and b?

Applying Skills

In 20–22, Mrs. Menendez uses computer software to record her checking account balance. Each time that she makes an entry, the amount that she enters is added to her balance.

20. If she writes a check for \$20, how should she enter this amount?

21. Mrs. Menendez had a balance of \$52 in her checking account and wrote a check for \$75.

a. How should she enter the \$75?

b. How should her new balance be recorded?

22. After writing the \$75 check, Mrs. Menendez realized that she would be overdrawn when the check was paid by the bank so she transferred \$100 from her savings account to her checking account. How should the \$100 be entered in her computer program?

1-2 WRITING AND SOLVING NUMBER SENTENCES

Equations

A sentence that involves numerical quantities can often be written in the symbols of algebra as an equation. For example, let x represent any number. Then the sentence "Three less than twice a number is 15" can be written as:

$$2x - 3 = 15$$

When we translate from one language to another, word order often must be changed in accordance with the rules of the language into which we are translating. Here we must change the word order for "three less than twice a number" to match the correct order of operations.

The **domain** is the set of numbers that can replace the variable in an algebraic expression. A number from the domain that makes an equation true is a **solution** or **root** of the equation. We can find the solution of an equation by writing a sequence of **equivalent equations**, or equations that have the same solution set, until we arrive at an equation whose solution set is evident. We find equivalent equations by changing both sides of the given equation in the same way. To do this, we use the following properties of equality:

Properties of Equality

- **Addition Property of Equality:** If equals are added to equals, the sums are equal.
- **Subtraction Property of Equality:** If equals are subtracted from equals, the differences are equal.
- **Multiplication Property of Equality:** If equals are multiplied by equals, the products are equal.
- **Division Property of Equality:** If equals are divided by non-zero equals, the quotients are equal.

On the left side of the equation $2x - 3 = 15$, the variable is multiplied by 2 and then 3 is subtracted from the product. We will simplify the left side of the equation by "undoing" these operations in reverse order, that is, we will first add 3 and then divide by 2. We can check that the number we found is a root of the given equation by showing that when it replaces x, it gives us a correct statement of equality.

$$\begin{aligned} 2x - 3 &= 15 \\ 2x - 3 + 3 &= 15 + 3 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

Check

$$\begin{aligned} 2x - 3 &= 15 \\ 2(9) - 3 &\stackrel{?}{=} 15 \\ 15 &= 15 \checkmark \end{aligned}$$

Often the definition of a mathematical term or a formula is needed to write an equation as the following example demonstrates:

EXAMPLE 1

Let $\angle A$ be an angle such that the complement of $\angle A$ is 6 more than twice the measure of $\angle A$. Find the measure of $\angle A$ and its complement.

Solution To write an equation to find $\angle A$, we must know that two angles are complements if the sum of their measures is 90°.

Let x = the measure of $\angle A$.

Then $2x + 6$ = the measure of the complement of $\angle A$.

The sum of the measures of an angle and of its complement is 90.

$$\begin{aligned} x \quad + \quad 2x + 6 &= 90 \\ 3x + 6 &= 90 \\ 3x &= 84 \\ x &= 28 \\ 2x + 6 &= 2(28) + 6 = 62 \end{aligned}$$

Therefore, the measure of $\angle A$ is 28 and the measure of its complement is 62.

Check The sum of the measures of $\angle A$ and its complement is $28 + 62$ or 90. ✔

Answer $m\angle A = 28$; the measure of the complement of $\angle A$ is 62.

EXAMPLE 2

Find the solution of the following equation: $|6x - 3| = 15$.

Solution Since $|15| = |-15| = 15$, the algebraic expression $6x - 3$ can be equal to 15 or to -15.

$$\begin{aligned} 6x - 3 &= 15 \\ 6x - 3 + 3 &= 15 + 3 \\ 6x &= 18 \\ x &= 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 6x - 3 &= -15 \\ 6x - 3 + 3 &= -15 + 3 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

Check: $x = 3$

$$\begin{aligned} |6x - 3| &= 15 \\ |6(3) - 3| &\stackrel{?}{=} 15 \\ |15| &= 15 \checkmark \end{aligned}$$

Check: $x = -2$

$$\begin{aligned} |6x - 3| &= 15 \\ |6(-2) - 3| &\stackrel{?}{=} 15 \\ |-15| &= 15 \checkmark \end{aligned}$$

Answer The solution set is $\{3, -2\}$.

Inequalities

A number sentence can often be an inequality. To find the solution set of an inequality, we use methods similar to those that we use to solve equations. We need the following two properties of inequality:

Properties of Inequality

- **Addition and Subtraction Property of Inequality:** If equals are added to or subtracted from unequals, the sums or differences are unequal in the same order.
- **Multiplication and Division Property of Inequality:** If unequals are multiplied or divided by positive equals, the products or quotients are unequal in the same order. If unequals are multiplied or divided by negative equals, the products or quotients are unequal in the opposite order.

EXAMPLE 3

Find all positive integers that are solutions of the inequality $4n + 7 < 27$.

Solution We solve this inequality by using a procedure similar to that used for solving an equation.

$$4n + 7 < 27$$
$$4n + 7 + (-7) < 27 + (-7)$$
$$4n < 20$$
$$n < 5$$

Since n is a positive integer, the solution set is $\{1, 2, 3, 4\}$. *Answer*

EXAMPLE 4

Polly has \$210 in her checking account. After writing a check for tickets to a concert, she has less than \$140 in her account but she is not overdrawn. If each ticket cost \$35, how many tickets could she have bought?

Solution Let x = the number of tickets.

The cost of x tickets, $35x$, will be subtracted from \$210, the amount in her checking account. Since she is not overdrawn after writing the check, her balance is at least 0 and less than \$140.

$$\begin{array}{rcl} 0 \leq & 210 - 35x < 140 & \\ \underline{-210} & \underline{-210 \qquad\quad -210} & \text{Add } -210 \text{ to each member of the inequality.} \\ -210 \leq & -35x < -70 & \\ \frac{-210}{-35} \geq & \frac{-35x}{-35} > \frac{-70}{-35} & \text{Divide each member of the inequality by } -35. \\ 6 \geq & x > 2 & \text{Note that dividing by a negative number reverses the order of the inequality.} \end{array}$$

Polly bought more than 2 tickets but at most 6.

Answer Polly bought 3, 4, 5, or 6 tickets.

Exercises

Writing About Mathematics

1. Explain why the solution set of the equation $12 - |x| = 15$ is the empty set.

2. Are $-4x > 12$ and $x > -3$ equivalent inequalities? Justify your answer.

Developing Skills

In 3–17, solve each equation or inequality. Each solution is an integer.

3. $5x + 4 = 39$ **4.** $7x + 18 = 39$ **5.** $3b + 18 = 12$

6. $12 - 3y = 18$ **7.** $9a - 7 = 29$ **8.** $13 - x = 15$

9. $|2x + 4| = 22$ **10.** $|3 - y| = 8$ **11.** $|4a - 12| = 16$

12. $|2x + 3| - 8 = 15$ **13.** $7a + 3 > 17$ **14.** $9 - 2b \leq 1$

15. $3 < 4x - 1 < 11$ **16.** $0 < x - 3 < 4$ **17.** $5 \geq 4b + 9 \geq 17$

Applying Skills

In 18–23, write and solve an equation or an inequality to solve the problem.

18. Peter had 156 cents in coins. After he bought 3 packs of gum he had no more than 9 cents left. What is the minimum cost of a pack of gum?

19. In an algebra class, 3 students are working on a special project and the remaining students are working in groups of five. If there are 18 students in class, how many groups of five are there?

20. Andy paid a reservation fee of \$8 plus \$12 a night to board her cat while she was on vacation. If Andy paid \$80 to board her cat, how many nights was Andy on vacation?

21. At a parking garage, parking costs \$5 for the first hour and \$3 for each additional hour or part of an hour. Mr. Kanesha paid \$44 for parking on Monday. For how many hours did Mr. Kanesha park his car?

22. Kim wants to buy an azalea plant for \$19 and some delphinium plants for \$5 each. She wants to spend less than \$49 for the plants. At most how many delphinium plants can she buy?

23. To prepare for a tennis match and have enough time for schoolwork, Priscilla can practice no more than 14 hours.. If she practices the same length of time on Monday through Friday, and then spends 4 hours on Saturday, what is the most time she can practice on Wednesday?

1-3 ADDING POLYNOMIALS

A **monomial** is a constant, a variable, or the product of constants and variables. Each algebraic expression, 3, a, ab, $-2a^2$, is a monomial.

Exponent

$$3a^4$$

Coefficient Base

A **polynomial** is the sum of monomials. Each monomial is a **term** of the polynomial. The expressions $3a^2 + 7a - 2$ is a polynomial over the set of integers since all of the numerical coefficients are integers. For any integral value of a, $3a^2 + 7a - 2$ has an integral value. For example, if $a = -2$, then:

$$\begin{aligned} 3a^2 + 7a - 2 &= 3(-2)^2 + 7(-2) - 2 \\ &= 3(4) + 7(-2) - 2 \\ &= 12 - 14 - 2 \\ &= -4 \end{aligned}$$

The same properties that are true for integers are true for polynomials: we can use the commutative, associative, and distributive properties when working with polynomials. For example:

$$\begin{aligned} (3a^2 + 5a) + (6 - 7a) &= (3a^2 + 5a) + (-7a + 6) && \text{Commutative Property} \\ &= 3a^2 + (5a - 7a) + 6 && \text{Associative Property} \\ &= 3a^2 + (5 - 7)a + 6 && \text{Distributive Property} \\ &= 3a^2 - 2a + 6 \end{aligned}$$

Note: When the two polynomials are added, the two terms that have the same power of the same variable factor are combined into a single term.

Two terms that have the same variable and exponent or are both numbers are called **similar terms** or **like terms**. The sum of similar terms is a monomial.

$3a^2 + 5a^2$	$-7ab + 3ab$	$x^3 + 4x^3$
$= (3 + 5)a^2$	$= (-7 + 3)ab$	$= (1 + 4)x^3$
$= 8a^2$	$= -4ab$	$= 5x^3$

Two monomials that are not similar terms cannot be combined. For example, $4x^3$ and $3x^2$ are not similar terms and the sum $4x^3 + 3x^2$ is not a monomial. A polynomial in simplest form that has two terms is a **binomial**. A polynomial in simplest form that has three terms is a **trinomial**.

Solving Equations and Inequalities

An equation or inequality often has a variable term on both sides. To solve such an equation or inequality, we must first write an equivalent equation or inequality with the variable on only one side.

For example, to solve the inequality $5x - 7 > 3x + 9$, we will first write an equivalent inequality that does not have a variable in the right side. Add the opposite of $3x$, $-3x$, to both sides. The terms $3x$ and $-3x$ are similar terms whose sum is 0.

$5x - 7 > 3x + 9$	
$-3x + 5x - 7 > -3x + 3x + 9$	Add $-3x$, the opposite of $3x$, to both sides.
$2x - 7 > 9$	$-3x + 3x = (-3 + 3)x = 0x = 0$
$2x - 7 + 7 > 9 + 7$	Add 7, the opposite of -7, to both sides.
$2x > 16$	Divide both sides by 2. Dividing by a positive does not reverse the inequality.
$x > 8$	

If x is an integer, then the solution set is $\{9, 10, 11, 12, 13, \ldots\}$.

EXAMPLE 1

a. Find the sum of $x^3 - 5x + 9$ and $x - 3x^3$.

b. Find the value of each of the given polynomials and the value of their sum when $x = -4$.

Solution **a.** The commutative and associative properties allow us to change the order and the grouping of the terms.

$$\begin{aligned}(x^3 - 5x + 9) + (x - 3x^3) &= (x^3 - 3x^3) + (-5x + x) + 9\\ &= (1 - 3)x^3 + (-5 + 1)x + 9\\ &= -2x^3 - 4x + 9 \text{ Answer}\end{aligned}$$

b.

$x^3 - 5x + 9$	$x - 3x^3$	$-2x^3 - 4x + 9$
$= (-4)^3 - 5(-4) + 9$	$= (-4) - 3(-4)^3$	$= -2(-4)^3 - 4(-4) + 9$
$= -64 + 20 + 9$	$= -4 + 192$	$= 128 + 16 + 9$
$= -35$ *Answer*	$= 188$ *Answer*	$= 153$ *Answer*

EXAMPLE 2

Subtract $(3b^4 + b + 3)$ from $(b^4 - 5b + 3)$ and write the difference as a polynomial in simplest form.

Solution Subtract $(3b^4 + b + 3)$ from $(b^4 - 5b + 3)$ by adding the opposite of $(3b^4 + b + 3)$ to $(b^4 - 5b + 3)$.

$$\begin{aligned}(b^4 - 5b + 3) - (3b^4 + b + 3) &= (b^4 - 5b + 3) + (-3b^4 - b - 3)\\ &= (b^4 - 3b^4) + (-5b - b) + (3 - 3)\\ &= -2b^4 - 6b \text{ Answer}\end{aligned}$$

EXAMPLE 3

Pam is three times as old as Jody. In five years, Pam will be twice as old as Jody. How old are Pam and Jody now?

Solution

Let x = Jody's age now
$3x$ = Pam's age now
$x + 5$ = Jody's age in 5 years
$3x + 5$ = Pam's age in 5 years

Pam's age in 5 years will be twice Jody's age in 5 years.

$$\begin{aligned}3x + 5 &= 2(x + 5)\\ 3x + 5 &= 2x + 10\\ -2x + 3x + 5 &= -2x + 2x + 10\\ x + 5 &= 0 + 10\\ x + 5 - 5 &= 10 - 5\\ x &= 5\\ 3x &= 15\end{aligned}$$

Answer Jody is 5 and Pam is 15.

Exercises

Writing About Mathematics

1. Danielle said that there is no integer that makes the inequality $|2x + 1| < x$ true. Do you agree with Danielle? Explain your answer.

2. A binomial is a polynomial with two terms and a trinomial is a polynomial with three terms. Jess said that the sum of a trinomial and binomial is always a trinomial. Do you agree with Jess? Justify your answer.

Developing Skills

In 3–12, write the sum or difference of the given polynomials in simplest form.

3. $(3y - 5) + (2y - 8)$

4. $(x^2 + 3x - 2) + (4x^2 - 2x + 3)$

5. $(4x^2 - 3x - 7) + (3x^2 - 2x + 3)$

6. $(-x^2 + 5x + 8) + (x^2 - 2x - 8)$

7. $(a^2b^2 - ab + 5) + (a^2b^2 + ab - 3)$

8. $(7b^2 - 2b + 3) - (3b^2 + 8b + 3)$

9. $(3 + 2b + b^2) - (9 + 5b + b^2)$

10. $(4x^2 - 3x - 5) - (3x^2 - 10x + 3)$

11. $(y^2 - y - 7) + (3 - 2y + 3y^2)$

12. $(2a^4 - 5a^2 - 1) + (a^3 + a)$

In 13–22, solve each equation or inequality. Each solution is an integer.

13. $7x + 5 = 4x + 23$

14. $y + 12 = 5y - 4$

15. $7 - 2a = 3a + 32$

16. $12 + 6b = 2b$

17. $2x + 3 < x + 15$

18. $5y - 1 \geq 2y + 5$

19. $9y + 2 \leq 7y$

20. $14c > 80 - 6c$

21. $(b - 1) - (3b - 4) = b$

22. $-3 - 2x \geq 12 + x$

Applying Skills

23. An online music store is having a sale. Any song costs 75 cents and any ringtone costs 50 cents. Emma can buy 6 songs and 2 audiobooks for the same price as 5 ringtones and 3 audiobooks. What is the cost of an audiobook?

24. The length of a rectangle is 5 feet more than twice the width.

a. If x represents the width of the rectangle, represent the perimeter of the rectangle in terms of x.

b. If the perimeter of the rectangle is 2 feet more than eight times the width of the rectangle, find the dimensions of the rectangle.

25. On his trip to work each day, Brady pays the same toll, using either all quarters or all dimes. If the number of dimes needed for the toll is 3 more than the number of quarters, what is the toll?

1-4 SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Absolute Value Equations

We know that if a is a positive number, then $|a| = a$ and that $|-a| = a$. For example, if $|x| = 3$, then $x = 3$ or $x = -3$ because $|3| = 3$ and $|-3| = 3$. We can use these facts to solve an **absolute value equation**, that is, an equation containing the absolute value of a variable.

For instance, solve $|2x - 3| = 17$. We know that $|17| = 17$ and $|-17| = 17$. Therefore $2x - 3$ can equal 17 or it can equal -17.

$$\begin{array}{rclcrcl} 2x - 3 &=& 17 & \qquad \text{or} \qquad & 2x - 3 &=& -17 \\ 2x - 3 + 3 &=& 17 + 3 & & 2x - 3 + 3 &=& -17 + 3 \\ 2x &=& 20 & & 2x &=& -14 \\ x &=& 10 & & x &=& -7 \end{array}$$

The solution set of $|2x - 3| = 17$ is $\{-7, 10\}$.

In order to solve an absolute value equation, we must first isolate the absolute value expression. For instance, to solve $|4a + 2| + 7 = 21$, we must first add -7 to each side of the equation to isolate the absolute value expression.

$$\begin{aligned} |4a + 2| + 7 &= 21 \\ |4a + 2| + 7 - 7 &= 21 - 7 \\ |4a + 2| &= 14 \end{aligned}$$

Now we can consider the two possible cases: $4a + 2 = 14$ or $4a + 2 = -14$

$$\begin{array}{rclcrcl} 4a + 2 &=& 14 & \qquad \text{or} \qquad & 4a + 2 &=& -14 \\ 4a + 2 - 2 &=& 14 - 2 & & 4a + 2 - 2 &=& -14 - 2 \\ 4a &=& 12 & & 4a &=& -16 \\ a &=& 3 & & a &=& -4 \end{array}$$

The solution set of $|4a + 2| + 7 = 21$ is $\{-4, 3\}$.

Note that the solution sets of the equations $|x + 3| = -5$ and $|x + 3| + 5 = 2$ are the empty set because absolute value is always positive or zero.

EXAMPLE 1

Find the solution of the following equation: $|4x - 2| = 10$.

Solution Since $|10| = |-10| = 10$, the algebraic expression $4x - 2$ can be equal to 10 or to -10.

$$\begin{aligned} 4x - 2 &= 10 & \quad\text{or}\quad && 4x - 2 &= -10 \\ 4x - 2 + 2 &= 10 + 2 & && 4x - 2 + 2 &= -10 + 2 \\ 4x &= 12 & && 4x &= -8 \\ x &= 3 & && x &= -2 \end{aligned}$$

Check: $x = 3$

$$\begin{aligned} |4x - 2| &= 10 \\ |4(3) - 2| &\stackrel{?}{=} 10 \\ |10| &= 10 \checkmark \end{aligned}$$

Check: $x = -2$

$$\begin{aligned} |4x - 2| &= 10 \\ |4(-2) - 2| &\stackrel{?}{=} 10 \\ |-10| &= 10 \checkmark \end{aligned}$$

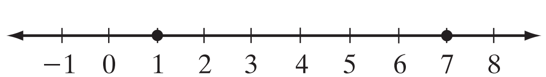
Answer The solution set is $\{3, -2\}$.

Absolute Value Inequalities

For any two given algebraic expressions, a and b, three relationships are possible: $a = b$, $a < b$, or $a > b$. We can use this fact to solve an **absolute value inequality** (an inequality containing the absolute value of a variable). For example, we know that for the algebraic expressions $|x - 4|$ and 3, there are three possibilities:

CASE 1 $|x - 4| = 3$

$$\begin{aligned} x - 4 &= -3 & \quad\text{or}\quad && x - 4 &= 3 \\ x &= 1 & \quad\text{or}\quad && x &= 7 \end{aligned}$$



Note that the solution set of this inequality consists of the values of x that are 3 units from 4 in either direction.

CASE 2 $|x - 4| < 3$

The solution set of this inequality consists of the values of x that are less than 3 units from 4 in either direction, that is, $x - 4$ is less than 3 and greater than -3.

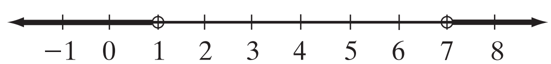
$$x - 4 > -3 \quad \text{and} \quad x - 4 < 3$$
$$x > 1 \quad \text{and} \quad x < 7$$

If x is an integer, the solution set is $\{2, 3, 4, 5, 6\}$. Note that these are the integers between the solutions of $|x - 4| = 3$.

CASE 3 $|x - 4| > 3$

The solution set of this inequality consists of the values of x that are more than 3 units from 4 in either direction, that is $x - 4$ is greater than 3 or less than -3.

$$x - 4 < -3 \quad \text{or} \quad x - 4 > 3$$
$$x < 1 \quad \text{or} \quad x > 7$$



If x is an integer, the solution set is $\{\ldots, -3, -2, -1, 0, 8, 9, 10, 11, \ldots\}$. Note that these are the integers that are less than the smaller solution of $|x - 4| = 3$ and greater than the larger solution of $|x - 4| = 3$.

We know that $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. We can use these relationships to solve inequalities of the form $|x| < k$ and $|x| > k$.

Solve $|x| < k$ for positive k

If $x \geq 0$, $|x| = x$.

Therefore, $x < k$ and $0 \leq x < k$.

If $x < 0$, $|x| = -x$.

Therefore, $-x < k$ or $x > -k$.

This can be written $-k < x < 0$.

The solution set of $|x| < k$ is

$$-k < x < k.$$

Solve $|x| > k$ for positive k

If $x \geq 0$, $|x| = x$.

Therefore, $x > k$.

If $x < 0$, $|x| = -x$.

Therefore, $-x > k$ or $x < -k$.

The solution set of $|x| > k$ is

$$x < -k \text{ or } x > k.$$

- **If $|x| < k$ for any positive number k, then $-k < x < k$.**
- **If $|x| > k$ for any positive number k, then $x > k$ or $x < -k$.**

EXAMPLE 2

Solve for b and list the solution set if b is an integer: $|6 - 3b| - 5 > 4$

Solution (1) Write an equivalent inequality with only the absolute value on one side of the inequality:

$$|6 - 3b| - 5 > 4$$
$$|6 - 3b| > 9$$

(2) Use the relationship derived in this section:

$$6 - 3b > 9 \quad \text{or} \quad 6 - 3b < -9$$

If $|x| > k$ for any positive number k, then $x > k$ or $x < -k$.

(3) Solve each inequality for b:

$6 - 3b > 9$	$6 - 3b < -9$
$-3b > 3$	$-3b < -15$
$\frac{-3b}{-3} < \frac{3}{-3}$	$\frac{-3b}{-3} > \frac{-15}{-3}$
$b < -1$	$b > 5$

Answer $\{\ldots, -5, -4, -3, -2, 6, 7, 8, 9, \ldots\}$

Exercises

Writing About Mathematics

1. Explain why the solution of $|-3b| = 9$ is the same as the solution of $|3b| = 9$.

2. Explain why the solution set of $|2x + 4| + 7 < 3$ is the empty set.

Developing Skills

In 3–14, write the solution set of each equation.

3. $|x - 5| = 12$
4. $|x + 8| = 6$
5. $|2a - 5| = 7$
6. $|5b - 10| = 25$
7. $|3x - 12| = 9$
8. $|4y + 2| = 14$
9. $|35 - 5x| = 10$
10. $|-5a| + 7 = 22$
11. $|8 + 2b| - 3 = 9$
12. $|2x - 5| + 2 = 13$
13. $|4x - 12| + 8 = 0$
14. $|7 - x| + 2 = 12$

In 15–26, solve each inequality and write the solution set if the variable is an element of the set of integers.

15. $|x| > 9$
16. $|y + 2| > 7$
17. $|b + 6| \leq 5$
18. $|x - 3| < 4$
19. $|y + 6| > 13$
20. $|2b - 7| \geq 9$
21. $|6 - 3x| < 15$
22. $|8 + 4b| \geq 0$
23. $|5 - b| + 4 < 9$
24. $|11 - 2b| - 6 > 11$
25. $|6 - 3b| + 4 < 3$
26. $|7 - x| + 2 \leq 12$

Applying Skills

27. A carpenter is making a part for a desk. The part is to be 256 millimeters wide plus or minus 3 millimeters. This means that the absolute value of the difference between the dimension of the part and 256 can be no more than 3 millimeters. To the nearest millimeter, what are the acceptable dimensions of the part?

28. A theater owner knows that to make a profit as well as to comply with fire regulations, the number of tickets that he sells can differ from 225 by no more than 75. How many tickets can the theater owner sell in order to make a profit and comply with fire regulations?

29. A cereal bar is listed as containing 200 calories. A laboratory tested a sample of the bars and found that the actual calorie content varied by as much as 28 calories. Write and solve an absolute value inequality for the calorie content of the bars.

1-5 MULTIPLYING POLYNOMIALS

We know that the product of any number of equal factors can be written as a power of that factor. For example:

$$a \times a \times a \times a = a^4$$

In the expression a^4, a is the **base**, 4 is the **exponent**, and a^4 is the **power**. The exponent tells us how many times the base, a, is to be used as a factor.

To multiply powers with like bases, keep the same base and add the exponents. For example:

$$x^3 \times x^2 = (x \times x \times x) \times (x \times x) = x^5$$
$$3^4 \times 3^5 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3) = 3^9$$

In general:

$$\boldsymbol{x^a \cdot x^b = x^{a+b}}$$

Note that we are not performing the multiplication but simply counting how many times the base is used as a factor.

Multiplying a Monomial by a Monomial

The product of two monomials is a monomial. We use the associative and commutative properties of multiplication to write the product.

$$\begin{aligned} 3a^2b(2abc) &= 3(2)(a^2)(a)(b)(b)(c) \\ &= 6a^3b^2c \end{aligned}$$

Note: When multiplying $(a^2)(a)$ and $(b)(b)$, the exponent of a and of b is 1.

The square of a monomial is the product of each factor of the monomial used twice as a factor.

$$\begin{array}{lll} (3ab^2)^2 & (-2x^3y)^2 & -2(x^3y)^2 \\ = (3ab^2)(3ab^2) & = (-2x^3y)(-2x^3y) & = -2(x^3y)(x^3y) \\ = 9a^2b^4 & = 4x^6y^2 & = -2x^6y^2 \end{array}$$

Multiplying a Polynomial by a Monomial

To multiply a monomial times a polynomial, we use the distributive property of multiplication over addition, $a(b + c) = ab + ac$:

$$\begin{array}{ll} -4(y - 7) & 5x(x^2 - 3x + 2) \\ = -4y - 4(-7) & = 5x(x^2) + 5x(-3x) + 5x(2) \\ = -4y + 28 & = 5x^3 - 15x^2 + 10x \end{array}$$

Note: The product of a monomial times a polynomial has the same number of terms as the polynomial.

Multiplying a Polynomial by a Binomial

To multiply a binomial by a polynomial we again use the distributive property of multiplication over addition. First, recall that the distributive property $a(b + c) = ab + ac$ can be written as:

$$(b + c)a = ba + ca$$

Now let us use this form of the distributive property to find the product of two binomials, for example:

$$\begin{aligned} (b + c) \quad (a) \quad &= b \quad (a) \quad + c \quad (a) \\ (x + 2)(x + 5) &= x(x + 5) + 2(x + 5) \\ &= x^2 + 5x + 2x + 10 \\ &= x^2 + 7x + 10 \end{aligned}$$

Multiplying two binomials (polynomials with two terms) requires four multiplications. We multiply each term of the first binomial times each term of the second binomial. The word **FOIL** helps us to remember the steps needed.

$$\begin{aligned} (x + 4)(x - 3) &= x(x - 3) + 4(x - 3) \\ & \quad\ \ \text{F} \qquad \text{O} \qquad \text{I} \qquad \text{L} \\ &= x^2 - 3x + 4x - 12 \\ &= x^2 + x - 12 \end{aligned}$$

Product of the **F**irst terms
Product of the **O**utside terms
Product of the **I**nside terms
Product of the **L**ast terms

Multiplying a Polynomial by a Polynomial

To multiply any two polynomials, multiply each term of the first polynomial by each term of the second. For example:

$$
\begin{aligned}
&(a^2 + a - 3)(2a^2 + 3a - 1) \\
&= a^2(2a^2 + 3a - 1) + a(2a^2 + 3a - 1) - 3(2a^2 + 3a - 1) \\
&= (2a^4 + 3a^3 - a^2) + (2a^3 + 3a^2 - a) - (6a^2 - 9a + 3) \\
&= 2a^4 + (3a^3 + 2a^3) + (-a^2 + 3a^2 - 6a^2) + (-a - 9a) + 3 \\
&= 2a^4 + 5a^3 - 4a^2 - 10a + 3
\end{aligned}
$$

Note: Since each of the polynomials to be multiplied has 3 terms, there are 3×3 or 9 products. After combining similar terms, the polynomial in simplest form has five terms.

EXAMPLE 1

Write each of the following as a polynomial in simplest form.

a. $ab(a^2 + 2ab + b^2)$ **b.** $(3x - 2)(2x + 5)$

c. $(y + 2)(y - 2)$ **d.** $(2a + 1)(a^2 - 2a - 2)$

Solution **a.** $ab(a^2 + 2ab + b^2) = a^3b + 2a^2b^2 + ab^3$ *Answer*

b.
$$
\begin{aligned}
(3x - 2)(2x + 5) &= 3x(2x + 5) - 2(2x + 5) \\
&= 6x^2 + 15x - 4x - 10 \\
&= 6x^2 + 11x - 10 \quad \textit{Answer}
\end{aligned}
$$

c.
$$
\begin{aligned}
(y + 2)(y - 2) &= y(y - 2) + 2(y - 2) \\
&= y^2 - 2y + 2y - 4 \\
&= y^2 - 4 \quad \textit{Answer}
\end{aligned}
$$

d.
$$
\begin{aligned}
(2a + 1)(a^2 - 2a - 2) &= 2a(a^2 - 2a - 2) + 1(a^2 - 2a - 2) \\
&= 2a^3 - 4a^2 - 4a + a^2 - 2a - 2 \\
&= 2a^3 - 3a^2 - 6a - 2 \quad \textit{Answer}
\end{aligned}
$$

EXAMPLE 2

Write in simplest form: $(2b)^2 + 5b[2 - 3(b - 1)]$

Solution (1) Simplify the innermost parentheses first:

$$\begin{aligned} & (2b)^2 + 5b[2 - 3(b - 1)] \\ &= (2b)^2 + 5b[2 - 3b + 3] \\ &= (2b)^2 + 5b[5 - 3b] \end{aligned}$$

(2) Multiply the terms in the brackets: $= (2b)^2 + 25b - 15b^2$

(3) Simplify powers: $= 4b^2 + 25b - 15b^2$

(4) Add similar terms: $= 25b - 11b^2$ *Answer*

EXAMPLE 3

Solve and check: $y(y + 2) - 3(y + 4) = y(y + 1)$

Solution (1) Simplify each side of the equation:

$$\begin{aligned} y(y + 2) - 3(y + 4) &= y(y + 1) \\ y^2 + 2y - 3y - 12 &= y^2 + y \\ y^2 - y - 12 &= y^2 + y \end{aligned}$$

(2) Add $-y^2$ to both sides of the equation:

$$\begin{aligned} -y^2 + y^2 - y - 12 &= -y^2 + y^2 + y \\ -y - 12 &= y \end{aligned}$$

(3) Add y to both sides of the equation:

$$\begin{aligned} y - y - 12 &= y + y \\ -12 &= 2y \end{aligned}$$

(4) Divide both sides of the equation by 2:

$$-6 = y$$

(5) *Check:*

$$\begin{aligned} y(y + 2) - 3(y + 4) &= y(y + 1) \\ -6(-6 + 2) - 3(-6 + 4) &\stackrel{?}{=} -6(-6 + 1) \\ -6(-4) - 3(-2) &\stackrel{?}{=} -6(-5) \\ 24 + 6 &\stackrel{?}{=} 30 \\ 30 &= 30 \checkmark \end{aligned}$$

Answer $y = -6$

Exercises

Writing About Mathematics

1. Melissa said that $(a + 3)^2 = a^2 + 9$. Do you agree with Melissa? Justify your answer.
2. If a trinomial is multiplied by a binomial, how many times must you multiply a monomial by a monomial? Justify your answer.

Developing Skills

In 3–23, perform the indicated operations and write the result in simplest form.

3. $2a^5b^2(7a^3b^2)$
4. $6c^2d(-2cd^3)$
5. $(6xy^2)^2$
6. $(-3c^4)^2$
7. $-(3c^4)^2$
8. $3b(5b - 4)$
9. $2x^2y(y - 2y^2)$
10. $(x + 3)(2x - 1)$
11. $(a - 5)(a + 4)$
12. $(3x + 1)(x - 2)$
13. $(a + 3)(a - 3)$
14. $(5b + 2)(5b - 2)$
15. $(a + 3)^2$
16. $(3b - 2)^2$
17. $(y - 1)(y^2 - 2y + 1)$
18. $(2x + 3)(x^2 + x - 5)$
19. $3a + 4(2a - 3)$
20. $b^2 + b(3b + 5)$
21. $4y(2y - 3) - 5(2 - y)$
22. $a^3(a^2 + 3) - (a^5 + 3a^3)$
23. $(z - 2)^3$

In 24–29, solve for the variable and check. Each solution is an integer.

24. $(2x + 1) + (4 - 3x) = 10$
25. $(3a + 7) - (a - 1) = 14$
26. $2(b - 3) + 3(b + 4) = b + 14$
27. $(x + 3)^2 = (x - 5)^2$
28. $4x(x + 2) - x(3 + 4x) = 2x + 18$
29. $y(y + 2) - y(y - 2) = 20 - y$

Applying Skills

30. The length of a rectangle is 4 more than twice the width, x. Express the area of the rectangle in terms of x.
31. The length of the longer leg, a, of a right triangle is 1 centimeter less than the length of the hypotenuse and the length of the shorter leg, b, is 8 centimeters less than the length of the hypotenuse.

    a. Express a and b in terms of c, the length of the hypotenuse.

    b. Express $a^2 + b^2$ as a polynomial in terms of c.

    c. Use the Pythagorean Theorem to write a polynomial equal to c^2.

1-6 FACTORING POLYNOMIALS

The **factors** of a monomial are the numbers and variables whose product is the monomial. Each of the numbers or variables whose product is the monomial is a factor of the monomial as well as 1 and any combination of these factors. For example, the factors of $3a^2b$ are 1, 3, a, and b, as well as $3a$, $3b$, a^2, ab, $3a^2$, $3ab$, a^2b, and $3a^2b$.

Common Monomial Factor

A polynomial can be written as a monomial times a polynomial if there is at least one number or variable that is a factor of each term of the polynomial. For instance:

$$4a^4 - 10a^2 = 2a^2(2a^2 - 5)$$

Note: $2a^2$ is the *greatest* **common monomial factor** of the terms of the polynomial because 2 is the greatest common factor of 4 and 10 and a^2 is the *smallest* power of a that occurs in each term of the polynomial.

EXAMPLE 1

Factor:

	Answers
a. $12x^2y^3 - 15xy^2 + 9y$	$= 3y(4x^2y^2 - 5xy + 3)$
b. $a^2b^3 + ab^2c$	$= ab^2(ab + c)$
c. $2x^2 - 8x + 10$	$= 2(x^2 - 4x + 5)$

Common Binomial Factor

We know that: $5ab + 3b = b(5a + 3)$

If we replace b by $(x + 2)$ we can write: $5a(x + 2) + 3(x + 2) = (x + 2)(5a + 3)$

Just as b is the common factor of $5ab + 3b$, $(x + 2)$ is the common factor of $5a(x + 2) + 3(x + 2)$. We call $(x + 2)$ the **common binomial factor**.

EXAMPLE 2

Find the factors of: $a^3 + a^2 - 2a - 2$

Solution Find the common factor of the first two terms and the common factor of the last two terms. Use the sign of the first term of each pair as the sign of the common factor.

$$\begin{aligned} a^3 + a^2 - 2a - 2 &= a^2(a + 1) - 2(a + 1) \\ &= (a + 1)(a^2 - 2) \quad \textit{Answer} \end{aligned}$$

Note: In the polynomial given in Example 2, the product of the first and last terms is equal to the product of the two middle terms: $a^3 \cdot (-2) = a^2 \cdot (-2a)$. This relationship will always be true if a polynomial of four terms can be factored into the product of two binomials.

Binomial Factors

We can find the binomial factors of a trinomial, if they exist, by reversing the process of finding the product of two binomials. For example:

$$\begin{aligned} (x + 3)(x - 2) &= x(x - 2) + 3(x - 2) \\ &= x^2 - 2x + 3x - 6 \\ &= x^2 + x - 6 \end{aligned}$$

Note that when the polynomial is written as the sum of four terms, the product of the first and last terms, $(x^2 \cdot -6)$, is equal to the product of the two middle terms, $(-2x \cdot 3x)$. We can apply these observations to factoring a trinomial into two binomials.

EXAMPLE 3

Factor $x^2 + 7x + 12$.

Solution **METHOD I**

(1) Write the trinomial as the sum of four terms by writing $7x$ as the sum of two terms whose product is equal to the product of the first and last terms:

$x^2 \cdot 12 = 12x^2$

$x \cdot 12x = 12x^2$ but $x + 12x \neq 7x$ ✘

$2x \cdot 6x = 12x^2$ but $2x + 6x \neq 7x$ ✘

$3x \cdot 4x = 12x^2$ and $3x + 4x = 7x$ ✔

(2) Rewrite the polynomial as the sum of four terms:

$$\begin{aligned} &x^2 + 7x + 12 \\ &= x^2 + 3x + 4x + 12 \end{aligned}$$

(3) Factor out the common monomial from the first two terms and from the last two terms:

$$= x(x + 3) + 4(x + 3)$$

(4) Factor out the common binomial factor:

$$= (x + 3)(x + 4)$$

METHOD 2

This trinomial can also be factored by recalling how the product of two binomials is found.

(1) The first term of the trinomial is the product of the first terms of the binomial factors:

$$x^2 + 7x + 12 = (x \qquad)(x \qquad)$$

(2) The last term of the trinomial is the product of the last terms of the binomial factors. Write all possible pairs of factors for which this is true.

$$\begin{array}{ll} (x + 1)(x + 12) & (x - 1)(x - 12) \\ (x + 2)(x + 6) & (x - 2)(x - 6) \\ (x + 3)(x + 4) & (x - 3)(x - 4) \end{array}$$

(3) For each possible pair of factors, find the product of the outside terms plus the product of the inside terms.

$$\begin{array}{ll} 12x + 1x = 13x \text{ ✘} & -12x + (-1x) = -13x \text{ ✘} \\ 6x + 2x = 8x \text{ ✘} & -6x + (-2x) = -8x \text{ ✘} \\ 3x + 4x = 7x \text{ ✔} & -3x + (-4x) = -7x \text{ ✘} \end{array}$$

(4) The factors of the trinomial are the two binomials such that the product of the outside terms plus the product of the inside terms equals $+7x$.

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Answer $(x + 3)(x + 4)$

EXAMPLE 4

Factor: $3x^2 - x - 4$

Solution

(1) Find the product of the first and last terms: $3x^2(-4) = -12x^2$

(2) Find the factors of this product whose sum is the middle term: $-4x + 3x = -x$

(3) Write the trinomial with four terms, using this pair of terms in place of $-x$:

$$3x^2 - x - 4$$
$$= 3x^2 - 4x + 3x - 4$$

(4) Factor the common factor from the first two terms and from the last two terms:

$$= x(3x - 4) + 1(3x - 4)$$

(5) Factor the common binomial factor:

$$= (3x - 4)(x + 1)$$

Answer $(3x - 4)(x + 1)$

Special Products and Factors

We know that to multiply a binomial by a binomial, we perform four multiplications. If all four terms are unlike terms, then the polynomial is in simplest form. Often, two of the four terms are similar terms that can be combined so that the product is a trinomial. For example:

$$\begin{aligned} &(a^2 + 3)(a - 2) \\ &= a^2(a - 2) + 3(a - 2) \\ &= a^3 - 2a^2 + 3a - 6 \end{aligned} \qquad \begin{aligned} &(x + 3)(x - 5) \\ &= x(x - 5) + 3(x - 5) \\ &= x^2 - 5x + 3x - 15 \\ &= x^2 - 2x - 15 \end{aligned}$$

When the middle terms are additive inverses whose sum is 0, then the product of the two binomials is a binomial.

$$\begin{aligned} (a + 3)(a - 3) &= a(a - 3) + 3(a - 3) \\ &= a^2 - 3a + 3a - 9 \\ &= a^2 + 0a - 9 \\ &= a^2 - 9 \end{aligned}$$

Therefore, the product of the sum and difference of the same two numbers is the difference of their squares. In general, the factors of the difference of two perfect squares are:

$$\mathbf{a^2 - b^2 = (a + b)(a - b)}$$

EXAMPLE 5

Factor:

	Think	*Write*
a. $4x^2 - 25$	$(2x)^2 - (5)^2$	$(2x + 5)(2x - 5)$
b. $16 - 9y^2$	$(4)^2 - (3y)^2$	$(4 + 3y)(4 - 3y)$
c. $36a^4 - b^4$	$(6a^2)^2 - (b^2)^2$	$(6a^2 + b^2)(6a^2 - b^2)$

When factoring a polynomial, it is important to make sure that each factor is a **prime polynomial** or has no factors other than 1 and itself. Once you have done this, the polynomial is said to be **completely factored**. For instance:

$$\begin{aligned} &3ab^2 - 6ab + 3a \\ &= 3a(b^2 - 2b + 1) \\ &= 3a(b - 1)(b - 1) \end{aligned} \qquad \begin{aligned} &x^4 - 16 \\ &= (x^2 + 4)(x^2 - 4) \\ &= (x^2 + 4)(x + 2)(x - 2) \end{aligned}$$

EXAMPLE 6

Factor: $5a^3 - a^2 - 5a + 1$

Solution (1) The product of the first and last terms is equal to the product of the two middle terms. Therefore, the polynomial is a product of two binomials:

$$5a^3 \cdot 1 \stackrel{?}{=} -a^2 \cdot -5a$$
$$5a^3 = 5a^3 \checkmark$$

(2) Find a common factor of the first two terms and then of the last two terms. Then, factor out the common binomial factor:

$$5a^3 - a^2 - 5a + 1$$
$$= a^2(5a - 1) - 1(5a - 1)$$
$$= (5a - 1)(a^2 - 1)$$

(3) The binomial factor $(a^2 - 1)$ is the difference of two squares, which can be factored into the sum and difference of the equal factors of the squares:

$$= (5a - 1)(a^2 - 1)$$
$$= (5a - 1)(a + 1)(a - 1)$$

Answer $(5a - 1)(a + 1)(a - 1)$

Exercises

Writing About Mathematics

1. Joel said that the factors of $x^2 + bx + c$ are $(x + d)(x + e)$ if $de = c$ and $d + e = b$. Do you agree with Joel? Justify your answer.

2. Marietta factored $x^2 + 5x - 4$ as $(x + 4)(x + 1)$ because $4(1) = 4$ and $4 + 1 = 5$. Do you agree with Marietta? Explain why or why not.

Developing Skills

In 3–8, write each polynomial as the product of its greatest common monomial factor and a polynomial.

3. $8x^2 + 12x$ **4.** $6a^4 - 3a^3 + 9a^2$ **5.** $5ab^2 - 15ab + 20a^2b$

6. $x^3y^3 - 2x^3y^2 + x^2y^2$ **7.** $4a - 12ab + 16a^2$ **8.** $21a^2 - 14a + 7$

In 9–26, write each expression as the product of two binomials.

9. $y(y + 1) - 1(y + 1)$ **10.** $3b(b - 2) - 4(b - 2)$ **11.** $2x(y + 4) + 3(y + 4)$

12. $a^3 - 3a^2 + 3a - 9$ **13.** $2x^3 - 3x^2 - 4x + 6$ **14.** $y^3 + y^2 - 5y - 5$

15. $x^2 + 7x + x + 7$ **16.** $x^2 + 5x + 6$ **17.** $x^2 - 5x + 6$

18. $x^2 + 5x - 6$ **19.** $x^2 - x - 6$ **20.** $x^2 + 9x + 20$

21. $3x^2 - 5x - 12$ **22.** $2y^2 + 5y - 3$ **23.** $5b^2 + 6b + 1$

24. $6x^2 - 13x + 2$ **25.** $4y^2 + 4y + 1$ **26.** $9x^2 - 12x + 4$

In 27–39, factor each polynomial completely.

27. $a^3 + 3a^2 - a - 3$

28. $5x^2 - 15x + 10$

29. $b^3 - 4b$

30. $4ax^2 + 4ax - 24a$

31. $12c^2 - 3$

32. $x^4 - 81$

33. $x^4 - 16$

34. $2x^3 + 13x^2 + 15x$

35. $4x^3 - 10x^2 + 6x$

36. $z^4 - 12z^2 + 27$

37. $(c + 2)^2 - 1$

38. $4 - (y - 1)^2$

39. $x^2y - 16y$

40. $3(x - 1)^2 - 12$

41. $9 - 9(x + 2)^2$

Applying Skills

In 42–45, each polynomial represents the area of a rectangle. Write two binomials that could represent the length and width of the rectangle.

42. $4x^2 - 7x - 2$

43. $16x^2 - 25$

44. $9x^2 - 6x + 1$

45. $3x^2 + 5x - 2$

1-7 QUADRATIC EQUATIONS WITH INTEGRAL ROOTS

An equation such as $3x + 4 = 16$ is a linear equation in one variable, that is, an equation in which the variable occurs to the first power only. An equation such as $x^2 - 3x + 2 = 0$ is a **quadratic equation** or a **polynomial equation of degree two** because the highest power of the variable is two. A quadratic equation is in **standard form** when it is written as a polynomial equal to 0. In general, if $a \neq 0$, the standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

To write the quadratic equation $3 + 2x(x - 1) = 5$ in standard form, first simplify the left member and then add -5 to each side of the equation.

$$\begin{aligned} 3 + x(x - 1) &= 5 \\ 3 + x^2 - x &= 5 \\ 3 + x^2 - x + (-5) &= 5 + (-5) \\ x^2 - x - 2 &= 0 \end{aligned}$$

Solving a Quadratic Equation

We know that $ab = 0$ if and only if $a = 0$ or $b = 0$. We can use this fact to solve a quadratic equation in standard form when the roots are integers. First, write the non-zero member of the equation as the product of factors, each of which contains the first power of the variable, and then set each factor equal to 0 to find the roots.

EXAMPLE 1

Solve the equation $3 + x(x - 1) = 5$.

Solution

$$
\begin{aligned}
3 + x(x - 1) &= 5 && \text{Write the equation in standard form.}\\
x^2 - x - 2 &= 0\\
(x - 2)(x + 1) &= 0 && \text{Factor the left side.}
\end{aligned}
$$

$$
\begin{array}{r|r}
x - 2 = 0 & x + 1 = 0\\
x = 2 & x = -1
\end{array}
\qquad \text{Set each factor equal to 0 and solve for } x.
$$

Check: $x = 2$

$$
\begin{aligned}
3 + x(x - 1) &= 5\\
3 + 2(2 - 1) &\stackrel{?}{=} 5\\
3 + 2(1) &\stackrel{?}{=} 5\\
3 + 2 &\stackrel{?}{=} 5\\
5 &= 5 \checkmark
\end{aligned}
$$

Check: $x = -1$

$$
\begin{aligned}
3 + x(x - 1) &= 5\\
3 + (-1)(-1 - 1) &\stackrel{?}{=} 5\\
3 - 1(-2) &\stackrel{?}{=} 5\\
3 + 2 &\stackrel{?}{=} 5\\
5 &= 5 \checkmark
\end{aligned}
$$

Answer $x = 2$ or -1

EXAMPLE 2

Solve for x: $2x^2 + 4x = 30$

Solution (1) Write the equation in standard form:

$$
\begin{aligned}
2x^2 + 4x &= 30\\
2x^2 + 4x - 30 &= 0
\end{aligned}
$$

(2) Factor the left member:

$$
\begin{aligned}
2(x^2 + 2x - 15) &= 0\\
2(x + 5)(x - 3) &= 0
\end{aligned}
$$

(3) Set each factor that contains the variable equal to zero and solve for x:

$$
\begin{array}{r|r}
x + 5 = 0 & x - 3 = 0\\
x = -5 & x = 3
\end{array}
$$

Answer $x = -5$ or 3

EXAMPLE 3

The length of a rectangle is 2 feet shorter than twice the width. The area of the rectangle is 84 square feet. Find the dimensions of the rectangle.

Solution Let w = the width of the rectangle.

$2w - 2$ = the length of the rectangle.

$$\text{Area} = \text{length} \times \text{width}$$
$$84 = (2w - 2)(w)$$
$$84 = 2w^2 - 2w$$
$$0 = 2w^2 - 2w - 84$$
$$0 = 2(w^2 - w - 42)$$
$$0 = 2(w - 7)(w + 6)$$
$$0 = w - 7 \quad \Big| \quad 0 = w + 6$$
$$7 = w \quad \Big| \quad -6 = w$$

The width must be a positive number. Therefore, only 7 feet is a possible width for the rectangle. When $w = 7$, $2w - 2 = 2(7) - 2 = 12$.

The area of the rectangle is $7(12) = 84$ square feet.

Answer The dimensions of the rectangle are 7 feet by 12 feet.

Exercises

Writing About Mathematics

1. Ross said that if $(x - a)(x - b) = 0$ means that $(x - a) = 0$ or $(x - b) = 0$, then $(x - a)(x - b) = 2$ means that $(x - a) = 2$ or $(x - b) = 2$. Do you agree with Ross? Explain why or why not.

2. If $(x - a)(x - b)(x - c) = 0$, is it true that $(x - a) = 0$, or $(x - b) = 0$ or $(x - c) = 0$? Justify your answer.

Developing Skills

In 3–17, solve and check each of the equations.

3. $x^2 - 4x + 3 = 0$ **4.** $x^2 - 7x + 10 = 0$ **5.** $x^2 - 5x - 6 = 0$

6. $x^2 + 6x + 5 = 0$ **7.** $x^2 + 10x - 24 = 0$ **8.** $x^2 - 9x = 10$

9. $4 - x(x - 3) = 0$ **10.** $x(x + 7) - 2 = 28$ **11.** $2x^2 - x = 12 + x$

12. $3x^2 - 5x = 36 - 2x$ **13.** $7 = x(8 - x)$ **14.** $9 = x(6 - x)$

15. $2x(x + 1) = 12$ **16.** $x(x - 2) + 2 = 1$ **17.** $3x(x - 10) + 80 = 5$

Applying Skills

18. Brad is 3 years older than Francis. The product of their ages is 154. Determine their ages.

19. The width of a rectangle is 12 feet less than the length. The area of the rectangle is 540 square feet. Find the dimensions of the rectangle.

20. The length of a rectangle is 6 feet less than three times the width. The area of the rectangle is 144 square feet. Find the dimensions of the rectangle.

21. The length of the shorter leg, a, of a right triangle is 6 centimeters less than the length of the hypotenuse, c, and the length of the longer leg, b, is 3 centimeters less than the length of the hypotenuse. Find the length of the sides of the right triangle.

22. The height h, in feet, of a golf ball shot upward from a ground level sprint gun is described by the formula $h = -16t^2 + 48t$ where t is the time in seconds. When will the ball hit the ground again?

1-8 QUADRATIC INEQUALITIES

DEFINITION

A **quadratic inequality** is an inequality that contains a polynomial of degree two.

When we solve a linear inequality, we use the same procedure that we use to solve a linear equation. Can we solve a *quadratic* inequality by using the same procedure that we use to solve a quadratic equation? How is the solution of the inequality $x^2 - 3x - 4 > 0$ similar to the solution of $x^2 - 3x - 4 = 0$?

To solve the equation, we factor the trinomial and write two equations in which each factor is equal to 0. To solve the inequality, can we factor the trinomial and write two inequalities in which each factor is greater than 0?

$$\begin{array}{rcr} x^2 - 3x - 4 = 0 & \qquad & x^2 - 3x - 4 > 0 \\ (x-4)(x+1) = 0 & & (x-4)(x+1) > 0 \\ x - 4 = 0 & & x - 4 \overset{?}{>} 0 \\ x + 1 = 0 & & x + 1 \overset{?}{>} 0 \end{array}$$

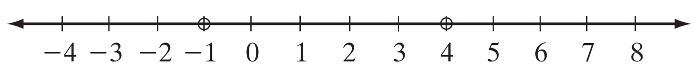
If the product of two factors is greater than 0, that is, positive, then it is true that each factor may be greater than 0 because the product of two positive numbers is positive. However, it is also true that each factor may be less than 0 because the product of two negative numbers is also positive. Therefore, when we solve a quadratic inequality, we must consider two possibilities:

$$x^2 - 3x - 4 > 0$$
$$(x - 4)(x + 1) > 0$$

$x - 4 > 0$ and $x + 1 > 0$	$x - 4 < 0$ and $x + 1 < 0$
$x > 4$ $x > -1$	$x < 4$ $x < -1$
If x is greater than 4 and greater than -1, then x is greater than 4.	If x is less than 4 and less than -1, then x is less than -1.

The solution set is $\{x : x > 4 \text{ or } x < -1\}$.

On the number line, the solutions of the equality $x^2 - 3x - 4 = 0$ are -1 and 4. These two numbers separate the number line into three intervals.



Choose a representative number from each interval:

Let $x = -3$:	**Let $x = 1$:**	**Let $x = 6$:**
$x^2 - 3x - 4 > 0$	$x^2 - 3x - 4 > 0$	$x^2 - 3x - 4 > 0$
$(-3)^2 - 3(-3) - 4 \overset{?}{>} 0$	$(1)^2 - 3(1) - 4 \overset{?}{>} 0$	$(6)^2 - 3(6) - 4 \overset{?}{>} 0$
$9 + 9 - 4 \overset{?}{>} 0$	$1 - 3 - 4 \overset{?}{>} 0$	$36 - 18 - 4 \overset{?}{>} 0$
$14 > 0$ ✔	$-6 \not> 0$ ✘	$14 > 0$ ✔

We find that an element from the interval $x < -1$ or an element from the interval $x > 4$ make the inequality true but an element from the interval $-1 < x < 4$ makes the inequality false.

A quadratic inequality in which the product of two linear factors is less than zero is also solved by considering two cases. A product is negative if the two factors have opposite signs. Therefore, we must consider the case in which the first factor is positive and the second factor is negative and the case in which the first factor is negative and the second factor is positive.

Procedure 1

To solve a quadratic inequality:

CASE 1 *The quadratic inequality is of the form* $(x - a)(x - b) > 0$

1. Let each factor be greater than 0 and solve the resulting inequalities.
2. Let each factor be less than 0 and solve the resulting inequalities.
3. Combine the solutions of the inequalities from steps 1 and 2 to find the solution set of the given inequality.

CASE 2 *The quadratic inequality is of the form* $(x - a)(x - b) < 0$

1. Let the first factor be greater than 0 and let the second factor be less than 0. Solve the resulting inequalities.
2. Let the first factor be less than 0 and let the second factor be greater than 0. Solve the resulting inequalities.
3. Combine the solutions of the inequalities from steps 1 and 2 to find the solution set of the given inequality.

A quadratic inequality can also be solved by finding the solutions to the corresponding equality. The solution to the inequality can be found by testing an element from each interval into which the number line is separated by the roots of the equality.

Procedure 2

To solve a quadratic inequality:

1. Find the roots of the corresponding equality.
2. The roots of the equality separate the number line into two or more intervals.
3. Test a number from each interval. An interval is part of the solution if the test number makes the inequality true.

EXAMPLE 1

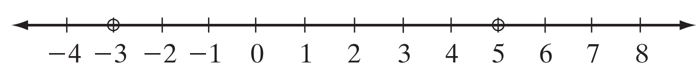
List the solution set of $x^2 - 2x - 15 < 0$ if x is an element of the set of integers.

Solution Factor the trinomial. One of the factors is negative and other is positive.

$$x^2 - 2x - 15 < 0$$
$$(x - 5)(x + 3) < 0$$

$x - 5 > 0$ and $x + 3 < 0$ $x > 5$ $\quad$ $x < -3$	$x - 5 < 0$ and $x + 3 > 0$ $x < 5$ $\quad$ $x > -3$
There are no values of x that are both greater than 5 and less than -3.	The solution set is $\{x : -3 < x < 5\}$.

Check The numbers -3 and 5 separate the number line into three intervals. Choose a representative number from each interval.



Let $x = -4$:

$$x^2 - 2x - 15 < 0$$
$$(-4)^2 - 2(-4) - 15 \stackrel{?}{<} 0$$
$$16 + 8 - 15 \stackrel{?}{<} 0$$
$$9 \not< 0 \text{ ✘}$$

Let $x = 1$:

$$x^2 - 2x - 15 < 0$$
$$(1)^2 - 2(1) - 15 \stackrel{?}{<} 0$$
$$1 - 2 - 15 \stackrel{?}{<} 0$$
$$-16 < 0 \text{ ✔}$$

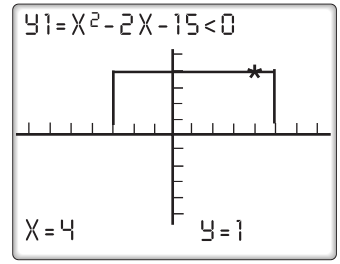
Let $x = 7$:

$$x^2 - 2x - 15 < 0$$
$$(7)^2 - 2(7) - 15 \stackrel{?}{<} 0$$
$$49 - 14 - 15 \stackrel{?}{<} 0$$
$$20 \not< 0 \text{ ✘}$$

When we choose a representative number from each of these intervals, we find that an element from the interval $x < -3$ and an element from the interval $x > 5$ make the inequality false but an element from the interval $-3 < x < 5$ makes the inequality true.

Answer $\{-2, -1, 0, 1, 2, 3, 4\}$

A graphing calculator can be used to verify the quadratic inequality of the examples. For instance, enter the inequality $x^2 - 2x - 15 < 0$ into Y_1. Use the **2nd** **TEST** menu to enter the inequality symbols $<, >, \leq$, and $\geq$. Using the **TRACE** button, we can then verify that the integers from -2 to 4 make the inequality true (the "Y = 1" in the bottom of the graph indicates that the inequality is true) while integers less than -2 or greater than 4 make the inequality false (the "Y = 0" in the bottom of the graph indicates that the inequality is false).



EXAMPLE 2

List the solution set of $x^2 + 6x + 8 \geq 0$ if x is an element of the set of integers.

Solution Factor the corresponding quadratic equality, $x^2 + 6x + 8 = 0$:

$$x^2 + 6x + 8 = 0$$
$$(x + 2)(x + 4) = 0$$

$$\begin{array}{r|r} x + 2 = 0 & x + 4 = 0 \\ x = -2 & x = -4 \end{array}$$

The roots -2 and -4 separate the number line into three intervals: $x < -4$, $-4 < x < -2$, and $-2 < x$. Test a number from each interval to find the solution of the inequality:

Let $x = -5$:

$$x^2 + 6x + 8 \overset{?}{\geq} 0$$
$$(-5)^2 + 6(-5) + 8 \overset{?}{\geq} 0$$
$$3 \geq 0 \checkmark$$

Let $x = -3$:

$$x^2 + 6x + 8 \overset{?}{\geq} 0$$
$$(-3)^2 + 6(-3) + 8 \overset{?}{\geq} 0$$
$$-1 \not\geq 0 \times$$

Let $x = 0$:

$$x^2 + 6x + 8 \overset{?}{\geq} 0$$
$$(0)^2 + 6(0) + 8 \overset{?}{\geq} 0$$
$$8 \geq 0 \checkmark$$

The inequality is true in the intervals $x < -4$ and $x > -2$. However, since the inequality is less than or equal to, the roots also make the inequality true. Therefore, the solution set is $\{x : x \leq -4 \text{ or } x \geq -2\}$.

Answer $\{\ldots, -6, -5, -4, -2, -1, 0, \ldots\}$

Exercises

Writing About Mathematics

1. Rita said that when the product of three linear factors is greater than zero, all of the factors must be greater than zero or all of the factors must be less than zero. Do you agree with Rita? Explain why or why not.

2. Shelley said that if $(x - 7)(x - 5) < 0$, then $(x - 7)$ must be the negative factor and $(x - 5)$ must be the positive factor.

   a. Do you agree with Shelly? Explain why or why not.

   b. When the product of two factors is negative, is it always possible to tell which is the positive factor and which is the negative factor? Justify your answer.

Developing Skills

In 3–17, write the solution set of each inequality if x is an element of the set of integers.

3. $x^2 + 5x + 6 < 0$
4. $x^2 + 5x - 6 > 0$
5. $x^2 - 3x + 2 \leq 0$
6. $x^2 - 7x + 10 > 0$
7. $x^2 - x - 6 < 0$
8. $x^2 - 8x - 20 \geq 0$
9. $x^2 + x - 12 < 0$
10. $x^2 - 6x + 5 > 0$
11. $x^2 - 2x \geq 0$
12. $x^2 - x < 6$
13. $x^2 - 4x + 4 > 0$
14. $x^2 - 4x + 4 \geq 0$
15. $x^2 + x - 2 < 0$
16. $2x^2 - 2x - 24 \leq 0$
17. $2x^2 - 2x - 24 > 0$

Applying Skills

18. A rectangular floor can be covered completely with tiles that each measure one square foot. The length of the floor is 1 foot longer than the width and the area is less than 56 square feet. What are the possible dimensions of the floor?

19. A carton is completely filled with boxes that are 1 foot cubes. The length of the carton is 2 feet greater than the width and the height of the carton is 3 feet. If the carton holds at most 72 cubes, what are the possible dimensions of the carton?

CHAPTER SUMMARY

The set of **natural numbers** is the set $\{1, 2, 3, 4, 5, 6, \ldots\}$. The set of **whole numbers** is the union of the set of natural numbers and the number 0. The set of **integers** is the union of set of whole numbers and their opposites.

The **absolute value** of a is symbolized by $|a|$. If $a > 0$, then $|a| = a - 0 = a$. If $a < 0$, then $|a| = 0 - a = -a$.

The **domain** is the set of numbers that can replace the variable in an algebraic expression. A number from the domain that makes an equation or inequality true is a **solution** or **root** of the equation or inequality.

We use the following properties of equality to solve an equation:

- **Addition Property of Equality:** If equals are added to equals, the sums are equal.
- **Subtraction Property of Equality:** If equals are subtracted from equals, the differences are equal.
- **Multiplication Property of Equality:** If equals are multiplied by equals, the products are equal.
- **Division Property of Equality:** If equals are divided by equals, the quotients are equal.

We use the following properties of inequality to solve inequalities:

- **Addition and Subtraction Property of Inequality:** If equals are added to or subtracted from unequals, the sums or differences are unequal in the same order.
- **Multiplication and Division Property of Inequality:** If unequals are multiplied or divided by positive equals, the products or quotients are unequal in the same order. If unequals are multiplied or divided by negative equals, the products or quotients are unequal in the opposite order.

A **monomial** is a constant, a variable, or the product of constants and variables. The **factors** of a monomial are the numbers and variables whose product is the monomial. A **polynomial** is the sum of monomials. Each monomial is a **term** of the polynomial.

An absolute value equation or inequality can be solved by using the following relationships:

- If $|x| = k$ for any positive number k, then $x = -k$ or $x = k$.
- If $|x| < k$ for any positive number k, then $-k < x < k$.
- If $|x| > k$ for any positive number k, then $x > k$ or $x < -k$.

If $a \neq 0$, the standard form of a quadratic equation is $ax^2 + bx + c = 0$. A quadratic equation that has integral roots can be solved by factoring the polynomial of the standard form of the equations and setting each factor that contains the variable equal to zero.

An inequality of the form $(x - a)(x - b) > 0$ can be solved by letting each factor be positive and by letting each factor be negative. An inequality of the form $(x - a)(x - b) < 0$ can be solved by letting one factor be positive and the other be negative.

VOCABULARY

1-1 Natural numbers • Counting numbers • Whole numbers • Opposite • Additive inverse • Integers • Commutative group • Absolute value

1-2 Domain • Solution • Root • Equivalent equations

1-3 Monomial • Polynomial • Term • Similar terms • Like terms • Binomial • Trinomial

1-4 Absolute value equation • Absolute value inequality

1-5 Base • Exponent • Power • FOIL

1-6 Factor • Common monomial factor • Common binomial factor • Prime polynomials • Completely factored

1-7 Quadratic equation • Polynomial equation of degree two • Standard form

1-8 Quadratic Inequality

REVIEW EXERCISES

In 1–12, write each expression is simplest form.

1. $5x - 7x$

2. $4(2a + 3) - 9a$

3. $2d - (5d - 7)$

4. $5(b + 9) - 3b(10 - b)$

5. $x(x + 3) - 4(5 - x)$

6. $8 - 2(a^2 + a + 4)$

7. $7d(2d + c) + 3c(4d - c)$

8. $(2x - 1)(3x + 1) - 5x^2$

9. $c^2 - (c + 2)(c - 2)$

10. $(2x + 1)^2 - (2x + 1)^2$

11. $(-2x)^2 - 2x^2$

12. $4y^2 + 2y(3y - 2) - (3y)^2$

In 13–24, factor each polynomial completely.

13. $2x^2 + 8x + 6$

14. $3a^2 - 30a + 75$

15. $5x^3 - 15x^2 - 20x$

16. $10ab^2 - 40a$

17. $c^4 - 16$

18. $3y^3 - 12y^2 + 6y - 24$

19. $x^3 + 5x^2 - x - 5$

20. $x^4 - 2x^2 - 1$

21. $2x^2 - 18x + 36$

22. $x^3 - 3x^2 + 2x$

23. $5a^4 - 5b^4$

24. $5x^2 + 22x - 15$

In 25–40, solve each equation or inequality for x. For each inequality, the solution set is a subset of the set of integers.

25. $8x + 27 = 5x$

26. $3(x - 7) = 5 + x$

27. $2x - 9 < 5x - 21$

28. $-3 \leq 2x - 1 < 7$

29. $|2x + 5| = 9$

30. $7 - |x + 1| = 0$

31. $|3 - 6y| + 2 > 11$

32. $4 - |x + 3| < 2$

33. $x^2 - 9x + 20 = 0$

34. $x(12 - x) = 35$

35. $x^2 + 7x + 6 < 0$

36. $x^2 - 2x - 35 > 0$

37. $x^2 \leq 5x$

38. $x(x + 3) > 0$

39. $4x^2 - 16x + 12 \leq 0$

40. $2x^2 + 2x - 4 \geq 0$

41. Explain why the equation $|3x - 5| + 4 = 0$ has no solution in the set of integers.

42. The length of a rectangle is 4 centimeters less than three times the width. The perimeter of the rectangle is 88 centimeters. What are the dimensions of the rectangle?

43. The length of a rectangle is 6 feet more than three times the width. The area of the rectangle is 240 square feet. What are the dimensions of the rectangle?

44. The length of the longer leg of a right triangle is 4 inches more than twice the length of the shorter leg. The length of the hypotenuse is 6 inches more than twice the length of the shorter leg. What are the lengths of the legs of the right triangle?

45. The equation $h = -16t^2 + 80t$ gives the height, h, in feet after t seconds when a ball has been thrown upward at a velocity of 80 feet per second.

a. Find the height of the ball after 3 seconds.

b. After how many seconds will the ball be at a height of 64 feet?

Exploration

A whole number that is the sum of all of its factors except itself is called a **perfect number**. Euclid said that if $(2^k - 1)$ is a prime, then $N = 2^{k-1}(2^k - 1)$ is a perfect number. A perfect number of this form is called a **Euclidean perfect number**.

1. Use the formula for a Euclidean perfect number to find the first four perfect numbers.

2. Show that a Euclidean perfect number is always even.

3. Show that a Euclidean perfect number must have 6 or 8 as the units digit. (*Hint:* What are the possible units digits of $(2^k - 1)$? Of 2^{k-1}?)